Assignment and Control of Two-Tiered Vehicle Traffic

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***Abstract*— This work considers the assignment of vehicle traffic consisting of both individual, opportunistic vehicles and a cooperative fleet of vehicles. The first set of vehicles seek a user-optimal policy and the second set seeks a fleet-optimal policy. We provide explicit sufficient conditions for the existence and uniqueness of a Nash equilibrium at which both policies are satisfied.**

We also propose two different algorithms to determine the equilibrium, one centralized and one decentralized. Furthermore, we present a control scheme to attain such an equilibrium in a dynamical network flow. An example is considered showing the workings of our scheme and numerical results are presented.

I. Introduction

The eventual introduction of autonomous vehicles onto roads will result in new possibilities for technological impact in traffic route planning. Already today, due to the number of connected devices and navigation solutions, many route decisions are made according to the solution of an optimization problem. While such schemes aim to optimize overall traffic flow and increase the utility for drivers and passengers, new challenges arise with the increase in optimal decision-making.

A classical example of optimal route planning decreasing utility is called Braess's Paradox. The paradox is that, given that each user tries to follow their own optimal path, the overall equilibrium may not be optimal from a system perspective, *i.e.*, if a system planner controls all users' choices, the planner can decrease overall delay in the system better than the collective of users. The loss of optimality here is often refereed to as price of anarchy [1].

In this work, we study the assignment of traffic where the network includes the operator of a large fleet of autonomous vehicles among many ordinary drivers. This situation may fast become a reality, as the tenth principle of the Shared Mobility Principles for Livable Cities [2] states,

> 10. We support that autonomous vehicles (AVS) in dense urban areas should be operated in only shared fleets.

The restriction to autonomous traffic to be operated solely by fleets would incentivize the need for novel methods of traffic control for use in networks with fleets.

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The traffic assignment problem for different classes of vehicles with different objectives has been studied previously. In [3], the authors study the equilibrium between user-optimal, *i.e.*, anarchistic, users and system-optimal users, *i.e.*, those who minimize the total delay for all users in system. In [4], the authors study existence and uniqueness of an equilibrium for three different classes of users: user-optimal, system-optimal, and fleet-optimal,[1] *i.e.*, those who minimize the delay for their own fleet. If one is able to control a certain fraction of vehicles, one can do so in such a way as to ensure system optimality of the equilibrium. This can be modeled as a Stackelberg game and in [5], a heuristic algorithm is introduced to compute the optimal actions which, when followed by enough of a proportion of vehicles, nearly results in system optimality.

In [6], the authors show that a user-optimal flow assignment can be implemented in distributed congestion-avoiding feedback controllers in a dynamical flow network for a single class of vehicles. In [7], the authors then show that those local routing polices can be updated through feedback of the given network state and introduce a sufficient condition for stability of the dynamical system according to these updates.

In this work, we study a two-tier assignment problem, where one class of users seeks user optimality, *i.e.*, minimizes their own travel time in the network, while the other class of users seeks fleet optimality, *i.e.*, minimizes the total travel time for the entire fleet. We provide simpler conditions for uniqueness of equilibrium than that of [4] and introduce two different algorithms to obtain the equilibrium, one centralized with proven convergence properties and the other decentralized. We also show the interaction between the two-tier assignment problem and a multi-commodity dynamical network flow model, with provable stability properties in the case of acyclic networks. By having this linkage between the static assignment problem and the dynamical network flow problems, we allow for further development of feedback controllers in order to improve the robustness of the assignment. Due to paper length limitations, we have omitted proofs of the latter results.

The rest of the paper is structured as follows. In Section II, we present the traffic model and optimization problems. In Section III, we study the assignment problem and present sufficient conditions for existence and uniqueness of the equilibrium along with two algorithms for computing the assignment. In Section IV, we show the relationship between delay functions and flow-density functions and propose a dynamical model. In Section VI, we present conclusions.

[1] Also called Nash-Cournot-optimal

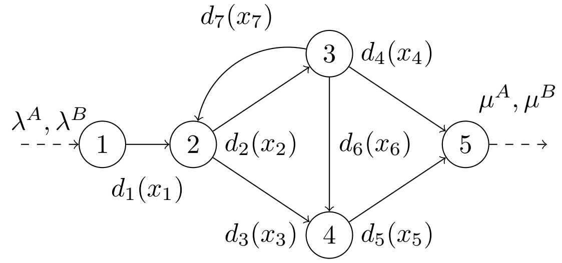

Fig. 1. Example of a traffic network

A. Notation

The set of real numbers is denoted by $\mathbb{R}$. We let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph, with node set $\mathcal{V}$ and link set $\mathcal{E}$. For a directed link $e = (i, j) \in \mathcal{E}$, let $\sigma(e) = i \in \mathcal{V}$ denote its tail and $\tau(e) = j \in \mathcal{V}$ its head. The set of incoming links to a node is given as $\mathcal{E}_v^- := \{e \in \mathcal{E} \mid \tau(e) = v\}$ and the set of outgoing links is $\mathcal{E}_v^+ := \{e =\in \mathcal{E} \mid \sigma(e) = v\}$. The norm $\|\cdot\|$ is the vector 2-norm.

II. Problem Formulation

A traffic network, modeled as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is a network of traffic flows along links $e \in \mathcal{E}$, representing unidirectional roads, and nodes $v \in \mathcal{V}$, representing junctions between roads; a schematic is shown in Fig. 1. The link set $\mathcal{E}$ is given as an ordered set of links $\mathcal{E} = \{e_i\}_{i=1}^{n_e}$ and the node set $\mathcal{V}$ is given as an ordered set of nodes $\mathcal{V} = \{v_j\}_{j=1}^{n_v}$. The graph topology, which defines the connections between nodes and links, can be represented by using the node-link incidence matrix $B \in \mathbb{R}^{n_v \times n_e}$ where,

$$B_{ji} = \begin{cases} 1 & \text{if } \sigma(e_i) = v_j \,, \\ -1 & \text{if } \tau(e_i) = v_j \,, \\ 0 & \text{o.w.} \end{cases}$$

In this work, we consider a traffic network facilitating two types of traffic. The first type, which we call Class A, consists of vehicles optimizing a user-optimal policy, *i.e.*, vehicles minimizing their marginal delay. The second type, which we call Class B, consists of vehicles optimizing a fleet-optimal policy, *i.e.*, vehicles minimizing the total delay of the fleet. The flows of Class A and Class B vehicles along each edge are denoted by $x^A, x^B \in \mathbb{R}^{n_e}$, respectively, where the entries of x^A and x^B are identified with the edges of $\mathcal{G}$ with the same ordering as the set $\mathcal{E}$. Exogenous inflows and outflows for Class A and Class B vehicles are given as $\lambda^A, \mu^A \in \mathbb{R}^{n_v}$ and $\lambda^B, \mu^B \in \mathbb{R}^{n_v}$, respectively, where the entries of λ^A, μ^A, λ^B, and μ^B are nonnegative and are identified with the nodes of $\mathcal{G}$ with the same ordering as the set $\mathcal{V}$. We further assume that the exogenous flows are feasible, *i.e.*, there exist $x^a, x^b \in \mathbb{R}^{n_e}$ such that $Bx^a = \lambda^A - \mu^A$ and $Bx^b = \lambda^B - \mu^B$.

Definition 1: Each link $e_i \in \mathcal{E}$ is identified with a *delay function* $d_i : \mathbb{R} \to \mathbb{R}$, mapping the flow x_i^A or x_i^B to a delay value. Delay functions are assumed to be twice continuously differentiable, strictly increasing, and nonnegative at 0.

Class A and Class B vehicles optimize different policies. Class A vehicles minimize the amount of time taken for each vehicle to arrive at its destination given the current state of traffic. In this case, the assignment of traffic flow along each link is given as the solution to the following optimization problem [8],

$$\min_{x^A} \quad g^A(x^A, x^B) := \sum_{i=1}^{n_e} \int_0^{x_i^A} d_i(s + x_i^B) \mathrm{d}s \,, \tag{1a}$$

$$\text{subject to} \quad Bx^A = \lambda^A - \mu^A \,, \tag{1b}$$

$$x^A \geq 0 \,. \tag{1c}$$

Class B vehicles minimize the average amount of time taken for each vehicle to arrive at its destination given the current state of traffic. In this case, the assignment of traffic flow along each link is given as the solution to the following optimization problem [8],

$$\min_{x^B} \quad g^B(x^A, x^B) := \sum_{i=1}^{n_e} x_i^B d_i(x_i^A + x_i^B) \,, \tag{2a}$$

$$\text{subject to} \quad Bx^B = \lambda^B - \mu^B \,, \tag{2b}$$

$$x^B \geq 0 \,. \tag{2c}$$

We study the assignment of traffic flow based on the solution of (1) and (2), the optimal behavior of vehicles when the dynamical traffic network is at equilibrium. Based on this assignment, it becomes necessary to route the traffic to achieve the desired flows. The assignment and routing scheme is presented in Fig. 2. In the figure, the assignment block computes the optimal traffic flows $x^{A*}, x^{B*} \in \mathbb{R}^{n_e}$ based on delay functions d_i and exogenous flows. Once the assignment is computed, the desired inflows are achieved by employing a routing policy which routes the fleet of Class B vehicles, denoted G^B, in the presence of a routing policy of the opportunistic behavior of Class A drivers, denoted G^A.

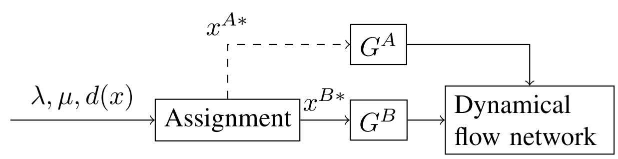

Fig. 2. Conceptual schematic of the assignment and control, note that Class A vehicles are assumed to find the optimal assignment on their own

III. Assignment

The solutions to (1) and (2) depend on each other, so it is of interest to investigate the properties of equilibrium values x^{A*} and x^{B*} which solve both equations simultaneously. In the following result, we provide sufficient conditions for the existence and uniqueness of an equilibrium, which depend solely on the properties of the delay functions d_i. The proof relies on results of [9], which provides conditions for existence and uniqueness of equilibria in convex games. The optimization problems (1) and (2) are both convex under certain choices of the delay functions d_i. Here, we present the requirements imposed on d_i which imply existence and uniqueness. We begin with a sufficient condition for existence.

Proposition 1: Assume that

$$2d_i'(x_i^A + x_i^B) + x_i^B d_i''(x_i^A + x_i^B) \geq 0, \tag{3}$$

for all $e_i \in \mathcal{E}$ and all feasible $x^A, x^B \in \mathbb{R}^{n_e}$. Then there exist x^{A*} and x^{B*} such that x^{A*} is a solution to (1) given $x^B = x^{B*}$, and x^{B*} is a solution to (2) given $x^A = x^{A*}$.

Proof: Theorem 1 of [9] states that there exists a solution to any concave (resp. convex) game. A game is concave (resp. convex) if the cost functions are concave (resp. convex) and individual constraints of every strategy are convex, *i.e.*, the constraints can be written in the form $h(x) \geq 0$ where every h is a convex function. Since the constraints are convex in both (1) and (2), all that is left to show is that the cost functions are both convex.

In the case of (1), the cost function is convex because d_i is monotonically increasing for all $i = 1, \ldots, n_e$, implying that its integral is monotonically increasing and therefore convex. In the case of (2), convexity follows from (3). ■

Results for uniqueness are less straightforward to obtain than results for existence. We proceed by providing a sufficient condition for uniqueness, which is a generalization of Proposition 4 of [4], in which the authors require $d_i(x_i)$ to be linear and strictly monotone.

Proposition 2: The pair x^{A*} and x^{B*} of Proposition 1 is unique if d_i satisfies the following relationship,

$$2d_i'(x_i^A + x_i^B) > x_i^B d_i''(x_i^A + x_i^B), \tag{4}$$

for all $e_i \in \mathcal{E}$ and feasible $x^A > 0$ and $x^B > 0$.

Proof: Let $S(x^A, x^B, y^A, y^B) = g^A(x^A, y^B) + g^B(y^A, x^B)$ and let,

$$h(x^A, x^B) = \begin{bmatrix} \nabla_{x^A} g^A(x^A, x^B) \\ \nabla_{x^B} g^B(x^A, x^B) \end{bmatrix},$$

be the pseudogradient of S. Since d_i is differentiable for any $i = 1, \ldots, n_e$, the pseudogradient is given by,

$$h(x^A, x^B) = \begin{bmatrix} d_1(x_1^A + x_1^B) \\ \vdots \\ d_n(x_n^A + x_n^B) \\ d_1(x_1^A + x_1^B) + x_1^B d_1'(x_1^A + x_1^B) \\ \vdots \\ d_n(x_n^A + x_n^B) + x_n^B d_n'(x_n^A + x_n^B) \end{bmatrix}.$$

Let H be the Jacobian of h. It is equal to,

$$H = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where,

$$\begin{aligned} A = B &= \operatorname{diag}\left(d_i'(x_i^A + x_i^B)\right)_{i=1}^{n_e}, \\ C &= \operatorname{diag}\left(d_i'(x_i^A + x_i^B) + x_i^B d_i''(x_i^A + x_i^B)\right)_{i=1}^{n_e}, \\ D &= \operatorname{diag}\left(2d_i'(y_i^A + x_i^B) + x_i^B d_i''(y_i^A + x_i^B)\right)_{i=1}^{n_e}. \end{aligned}$$

Now let,

$$F = H + H^T = \begin{bmatrix} 2A & A+C \\ A+C & 2D \end{bmatrix} = \begin{bmatrix} 2A & D \\ D & 2D \end{bmatrix}.$$

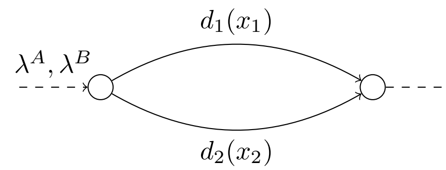

Fig. 3. The network of Example 1

According to Theorem 7 of [9], S is strictly diagonally convex if H is positive definite for all feasible $x^A > 0$ and x^B. Furthermore, according to Theorem 3 of [9], if S is strictly diagonally convex then the equilibrium x^{A*} and x^{B*} is unique. We now show that F is positive definite by considering its Schur complement $\bar{F} = 2D + D^{\mathrm{T}}(2A)^{-1}D = 2D + (2A)^{-1}D^2$. The Jacobian is positive definite if and only if $\bar{F}$ and A are positive definite. The matrix A is positive definite because d_i is strictly increasing. The matrix $\bar{F}$ is,

$$\begin{aligned} \bar{F} = \operatorname{diag}\Bigg(&4d_i'(x_i^A + x_i^B) + 2x_i^B d_i''(x_i^A + x_i^B) \\ &- \frac{(2d_i'(x_i^A + x_i^B) + x_i^B d_i''(x_i^A + x_i^B))^2}{2d_i'(x_i^A + x_i^B)}\Bigg)_{i=1}^{n_e}, \\ = \operatorname{diag}&\left(2d_i'(x_i^A + x_i^B) - \frac{(x_i^B d_i''(x_i^A + x_i^B))^2}{2d_i'(x_i^A + x_i^B)}\right)_{i=1.}^{n_e}, \end{aligned}$$

and its elements are positive when (4) is satisfied. ■

We now consider the special case where the delay functions are given by a linear relation to the power of a positive exponent. Let,

$$d_i(x_i) = (\alpha_i x_i + \beta_i)^{c_i}, \tag{5}$$

where $\alpha_i > 0$, $\beta_i \geq 0$ and $c_i > 0$ are parameters. We introduce this form of delay function because it is useful in modeling the behavior of traffic flow in the free-flow regime, where the flows do not saturate. In the sequel, we will show how these delay functions can recover the free-flow regime of the fundamental diagram.

Corollary 1: Suppose the delay functions take the form in (5) with $\alpha_i, c_i > 0$ and $\beta_i \geq 0$ for all $e_i \in \mathcal{E}$. The solution exists and is unique if $c_i \leq 3$.

We have thus far shown that an equilibrium solution to the optimization problems (1) and (2) always exists and given conditions that guarantee this equilibrium to be unique. We now present a case in which a slight modification no longer guarantees uniqueness of the equilibrium solution. Specifically, if we introduce a capacity constraint on the flow of each node, then uniqueness may not be guaranteed since capacity constraints are concave constraints.

Example 1: Consider the network of Fig. 3 with $d_1(x_1) = x_1$ and $d_2(x_2) = 2x_2$ and impose a constraint on the flow capacity of link 1 so that $x_1^A + x_1^B \leq 1$. Then $x^A = (0, 1)$, $x^B = (1, 0)$ is an equilibrium because the minimizer of $x_1^B d_1(x_1^B) + x_2^B d_2(1 + x_2^B) = x_1^{B2} + 2x_2^B(1 - x_2^B)$ is $(1, 0)$ and the minimizer of the other optimization must satisfy $0 \leq x_1^A + x_1^B = x_1^A + 1 \leq 1$, implying that $x_1^A = 0$ and therefore $x^A = (0, 1)$. Furthermore, $x^A = (1, 0)$,

$x^B = (0,1)$ is also an equilibrium because the minimizer of $\int_0^{x_1^A} d_1(s)ds + \int_0^{x_2^A} d_2(s+1)\mathrm{d}s = \frac{1}{2}x_1^{A2} + x_2^{A2} + 2x_2^A$ is $(1,0)$ and the minimizer of the other optimization must satisfy $0 \le x_1^A + x_1^B = 1 + x_1^B \le 1$, implying that $x_1^B = 0$ and therefore $x^B = (0,1)$.

In the sequel we present two algorithms, one centralized and one decentralized, to determine traffic equilibrium.

A. Algorithms for determining traffic assignment

We propose two algorithms for determining the equilibrium flows x^{A*} and x^{B*}. The first is a centralized algorithm based on the idea that the optimal flow assignment for each class of vehicles are updated simultaneously in the constraint-admissible, opposite direction of the gradient of the objective functions. This is done by keeping the total flow in the system the same as an initial assignment, while making sure that the flow on each link remains non-negative.

1) Centralized algorithm: We begin by defining the Lagrangian corresponding to the optimization problem (1),

$$L^A(x^A, x^B, \gamma^A, \kappa^A) = -g^A(x^A, x^B) + (\gamma^A)^T(Bx^A - \lambda^A + \mu^A) + (\kappa^A)^T x^A ,$$

where $\gamma^A \in \mathbb{R}^{n_v}$ and $\kappa^A \in \mathbb{R}^{n_e}$. We define L^B analogously,

$$L^B(x^A, x^B, \gamma^B, \kappa^B) = -g^B(x^A, x^B) + (\gamma^B)^T(Bx^B - \lambda^B + \mu^B) + (\kappa^B)^T x^B .$$

The algorithm we propose updates the flow assignments by following the gradients of the Lagrangians,

$$\begin{aligned} \dot{x}^A &= \nabla_{x^A} L^A = -\nabla_{x^A} g^A(x^A, x^B) + B^{\mathrm{T}}\gamma^A + \kappa^A, \\ &=: f^A(x^A, x^B, \gamma^A, \kappa^A) , \qquad (6) \\ \dot{x}^B &= \nabla_{x^B} L^B = -\nabla_{x^B} g^B(x^A, x^B) + B^{\mathrm{T}}\gamma^B + \kappa^B, \\ &=: f^B(x^A, x^B, \gamma^B, \kappa^B) . \qquad (7) \end{aligned}$$

The values of γ^A, γ^B, κ^A, and κ^B are chosen according to the optimization,

$$(\gamma^A, \kappa^A) = \underset{\kappa^A \ge 0}{\arg\min} \, \|f^A(x^A, x^B, \gamma^A, \kappa^A)\|^2 , \qquad (8)$$

$$(\gamma^B, \kappa^B) = \underset{\kappa^B \ge 0}{\arg\min} \, \|f^B(x^A, x^B, \gamma^B, \kappa^B)\|^2 , \qquad (9)$$

subject to the constraint that $\kappa_i^A = 0$ and $\kappa_i^B = 0$ whenever $x_i^A > 0$ and $x_i^B > 0$, respectively.

Proposition 3: Given feasible initial states $x^A(0)$, $x^B(0)$, *i.e.*, $Bx^k(0) = \lambda^k - \mu^k$ and $x^k(0) \ge 0$ are satisfied for $k = A, B$, the algorithm (6)-(9) converges to an equilibrium.

2) Decentralized algorithm: We now present a decentralized algorithm for obtaining traffic assignment equilibrium. The benefits of decentralized algorithms are several. Since traffic networks are often large-scale networks, we are able to decrease the amount of computations by solving part of the optimization locally in the network and restricting communication to the local neighborhood. Furthermore, it imposes desired scalability problems to the algorithm. If a part of the network topology changes, only the algorithm associated with the neighboring areas needs to be updated. The algorithm presented above is in general not decentralized due to the fact that γ in (8)–(9) is computed using $(\bar{B}^T\bar{B})^{-1}\bar{B}^T$, which is in general not limited to having entries between neighboring nodes. Furthermore, when one of the links has zero flow, the κ value associated with it may affect the computations of γ all over the network. For our decentralized algorithm, we propose a dual descent scheme. We utilize the idea that, in order to compute the optimal flow on one link $e_i \in \mathcal{E}$, we only need information about the Lagrange multipliers $\gamma^{A,B}$ associated with the tail node $\tau(e_i)$ and head node $\sigma(e_i)$. The dynamics to update the Lagrange multiplier for each node $v_j \in \mathcal{V}$ are then only dependent on the incoming flows $x_i^{A,B}$ for the links satisfying $\sigma(e_i) = v_j$ and the outgoing flows $x_i^{A,B}$ for the links satisfying $\tau(e_i) = v_j$. The algorithm consists of solving the following,

$$\begin{aligned} x^A &= \underset{x^A \ge 0}{\arg\min} \, L^A(x^A, x^B, \gamma^A, 0) , \\ x^B &= \underset{x^B \ge 0}{\arg\min} \, L^B(x^A, x^B, \gamma^B, 0) , \\ \dot{\gamma}^A &= Bx^A - \lambda^A + \mu^A , \\ \dot{\gamma}^B &= Bx^B - \lambda^B + \mu^B . \end{aligned}$$

Since x^A and x^B depend only on the differences between different components in γ^A and γ^B, the system above will have multiple equilibria in γ^A and γ^B. These will be related to each other by a constant offset term so that if $(\tilde{\gamma}^A, \tilde{\gamma}^B)$ is an equilibrium, then $(\tilde{\gamma}^A + c\mathbf{1}_{n_e}, \tilde{\gamma}^B + d\mathbf{1}_{n_e})$ will be an equilibrium for any scalars c and d. This is not a limitation, however, due to the fact the difference in Lagrange multipliers determines the flow.

The decentralized aspect of the algorithm can be shown by deriving the necessary conditions,

$$\begin{aligned} 0 &= d_i(x_i^A + x_i^B) + \gamma_{\tau(i)}^A - \gamma_{\sigma(i)}^A , \\ 0 &= x_i^B d_i'(x_i^A + x_i^B) + d_i(x_i^A + x_i^B) + \gamma_{\tau(i)}^B - \gamma_{\sigma(i)}^B , \end{aligned}$$

for all $i = 1, \ldots, n_e$. Defining $\tilde{d}^{-1}$ as,

$$\tilde{d}^{-1}(\gamma_i) = \begin{cases} 0 & \text{if } \gamma_i < d_i(0), \\ d_i^{-1}(\gamma_i) & \text{o.w.} \end{cases} \qquad (10)$$

Then,

$$\begin{aligned} x_i^A &= \max\left\{ \tilde{d}_i^{-1}(\Gamma_i^A) - x_i^B, 0 \right\} , \\ x_i^B &= \max\left\{ \frac{\Gamma_i^B - \Gamma_i^A}{d_i'(\tilde{d}^{-1}(\Gamma_i^A))}, 0 \right\} , \end{aligned}$$

where $\Gamma_i^k = \gamma_{\sigma(i)}^k - \gamma_{\tau(i)}^k$ for $k = A, B$, thus showing that updates of the flow are independent of nonadjacent nodes.

Example 2: In the case when the delay functions are affine, *i.e.*, $d_i(x_i) = \alpha_i x_i + \beta_i$, the expressions introduced above become,

$$\begin{aligned} x_i^A &= \max\left\{ \frac{\Gamma_i^A - \beta_i}{\alpha_i} - x_i^B, 0 \right\} , \\ x_i^B &= \max\left\{ \frac{\Gamma_i^B - \Gamma_i^A}{\alpha_i}, 0 \right\} . \end{aligned}$$

IV. Routing of Two-Tier Traffic

In this section, we study the stability of a routing scheme that statically allocates traffic to each node dependent on the solution to the assignment problem. To model the flow dynamics in the network, we introduce traffic density vectors $\rho^A, \rho^B \in \mathbb{R}^{n_e}$. A static relationship between the densities and the flows on each link can be derived from delay functions, in the same way as proposed for one class of vehicles in [6]. Specifically, since the outflow on each link is given by the (average) speed times the density ρ_i, and the the speed is given by the length of the link $\ell_i > 0$ over the delay $d_i(x_i)$, the following relationship is assumed to hold,

$$\rho_i = \frac{x_i d_i(x_i)}{\ell_i}\,.$$

We define the flow function W_i to be the map from density ρ_i to flow x_i, *i.e.*, $x_i = W_i(\rho_i)$. Using the above relationship, we obtain,

$$d_i(x_i) = \begin{cases} \frac{W_i^{-1}(x_i)\ell_i}{x_i} & \text{if } x_i > 0\,, \\ \frac{\ell_i}{W_i'(0)} & \text{if } x_i = 0\,. \end{cases}$$

We observe that the flow function $W_i(\rho_i)$ is always strictly increasing, due to the fact that

$$W_i'(W_i^{-1}(x_i)) = \frac{\ell_i}{d_i(x_i) + x_i d_i'(x_i)} > 0\,.$$

Example 3: Let the delay function be given by $d_i(x_i) = \alpha_i x_i + \beta_i$. Then,

$$\rho_i = \frac{x_i(\alpha_i x_i + \beta_i)}{\ell_i}\,,$$

and hence,

$$x_i = W_i(\rho_i) = \sqrt{\frac{1}{4}\left(\frac{\beta_i}{\alpha_i}\right)^2 + \frac{\rho_i \ell_i}{\alpha_i}} - \frac{1}{2}\frac{\beta_i}{\alpha_i}\,.$$

The density dynamics follows from the conservation of mass, i.e., the change of density for each class of vehicles on each link is equal to the inflow minus the outflow. Combined with the static relationship between flow and density, the dynamic equations become,

$$\dot{\rho}_i^A = \tilde{\lambda}^A_{\sigma(e_i)} G^A_{\sigma(e_i)\to i} - \frac{\rho_i^A}{\rho_i} W_i(\rho_i)\,, \tag{11a}$$

$$\dot{\rho}_i^B = \tilde{\lambda}^B_{\sigma(e_i)} G^B_{\sigma(e_i)\to i} - \frac{\rho_i^B}{\rho_i} W_i(\rho_i)\,, \tag{11b}$$

where,

$$\tilde{\lambda}_v^A = \sum_{e_i \in \mathcal{E}_v^-} \frac{\rho_i^A}{\rho_i} W_i(\rho_i) + \lambda_v^A\,, \tag{12a}$$

$$\tilde{\lambda}_v^B = \sum_{e_i \in \mathcal{E}_v^-} \frac{\rho_i^B}{\rho_i} W_i(\rho_i) + \lambda_v^B\,, \tag{12b}$$

and $G^k_{v\to i}$, $k = A, B$, is the fraction of flow of class A or B that should be routed from node v to link e_i. For a given flow assignment x^{k*}, the routing policies are given by,

$$G^k_{v\to i} := \begin{cases} \frac{x_i^{k*}}{\sum_{e_j\in\mathcal{E}_v^+} x_j^{k*}} & \text{if } \sum_{e_j\in\mathcal{E}_v^+} x_j^{k*} > 0\,, \\ 0 & \text{o.w.} \end{cases} \tag{13}$$

The following result is similar to the results of [10], with a modification made to take unbounded W_i into account.

Proposition 4: Suppose $\mathcal{G}$ is acyclic. Then the dynamics given by (11)-(13) converge to the assigned equilibrium.

V. Numerical Example

In this section, we perform numerical simulations so that we may test the schemes proposed in Sections III and IV on a network with dynamics given in (11). The network, shown in Fig. 1, is similar to the one used to illustrate Braess's paradox, but with one additional link, which we have added to test if our algorithm can handle cycles, we have added an extra link to the Braess network.

Depending on the choice of parameters, the user- and fleet-optimal assignments can be different. Moreover, the optimal assignment in the first node can depend on the delay functions in other parts on the network, rather than just the outgoing links from the first node. Let the delay functions be affine, given by $d_i(x_i) = \alpha_i x_i + \beta_i$ for all links with the values of α_i and β_i specified in Table I. We let $\lambda^A = \mu^A = 1$ and $\lambda^B = \mu^B = 4$ and, for simplicity, we let all the links be of unit length, *i.e.*, $\ell_i = 1$ for all links. The results of simulations using the centralized and decentralized algorithm are shown in Fig. 4. In both simulations, the solver dynamics are simulated by using an Euler solver with a step length of 0.1. While the decentralized algorithm does not require a start from a feasible solution, *i.e.*, the initial flows x^A and x^B can be any non-negative value, the centralized algorithm needs to start from a feasible solution because the algorithm has no information about exogenous arrivals.

The routing polices G^A and G^B, presented in (13), are determined according to the desired assignment. Fig. 5 presents the outflows of each class of vehicles from each link. Two simulations are performed. The first simulation corresponds to setting all initial densities to zero, *i.e.*, $\rho_i^A(0) = \rho_i^B(0) = 0$ for all $e_i \in \mathcal{E}$. The second simulation corresponds to setting all initial densities to 5, *i.e.*, $\rho_i^A(0) = \rho_i^B(0) = 5$ for all $e_i \in \mathcal{E}$. In the first simulation, link e_7 can be removed from the network without causing any effect on the dynamics. This is because $G^k_{v_3\to e_7} = 0$ for $k = A, B$, thus making the network equivalent to an acyclic network with proven convergence properties according to Proposition 4. In the second simulation, link e_7 contributes with a converging inflow to node v_1 but has no inflow to itself. Therefore it can be seen as a converging inflow to node v_1 and the proof of Proposition 4 can be slightly modified to show convergence in this case as well.

VI. Conclusion

In this paper, we have analyzed the assignment and control of vehicle traffic in a network of vehicles that follow either user-optimal or fleet-optimal paths. For the assignment problem, we have provided sufficient conditions for the existence and uniqueness of an equilibrium, as well as two algorithms to compute it. For control, we have showed how the assignment may be achieved using static routing.

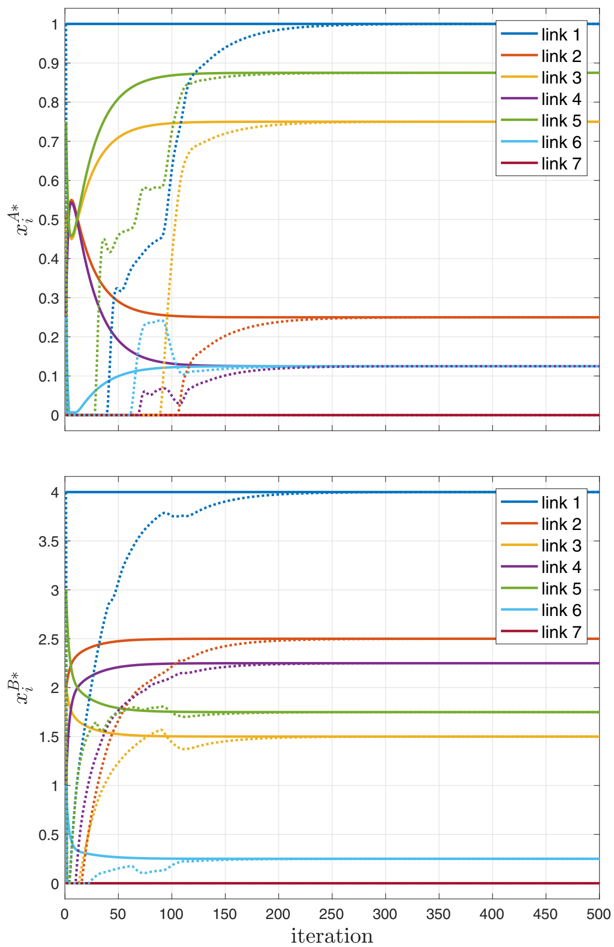

Fig. 4. The assigned equilibria x^{A*} and x^{B*} computed by centralized (solid) and decentralized (dotted) algorithms, plotted as a function of algorithm iteration. Both algorithms converge to the unique equilibrium, but the centralized one converges in fewer iterations.

TABLE I

PARAMETERS AND ASSIGNED FLOWS FOR SIMULATIONS

e_i	α_i	β_i	x_i^{A*}	x_i^{B*}
e_1	1	1	1	4
e_2	1	1	0.25	2.5
e_3	2	1	0.75	1.5
e_4	1	3	0.125	2.25
e_5	1	1	0.875	1.75
e_6	2	1	0.125	0.25
e_7	1	2	0	0

We applied our scheme to a modified Braess network and presented numerical results that exemplify our approach.

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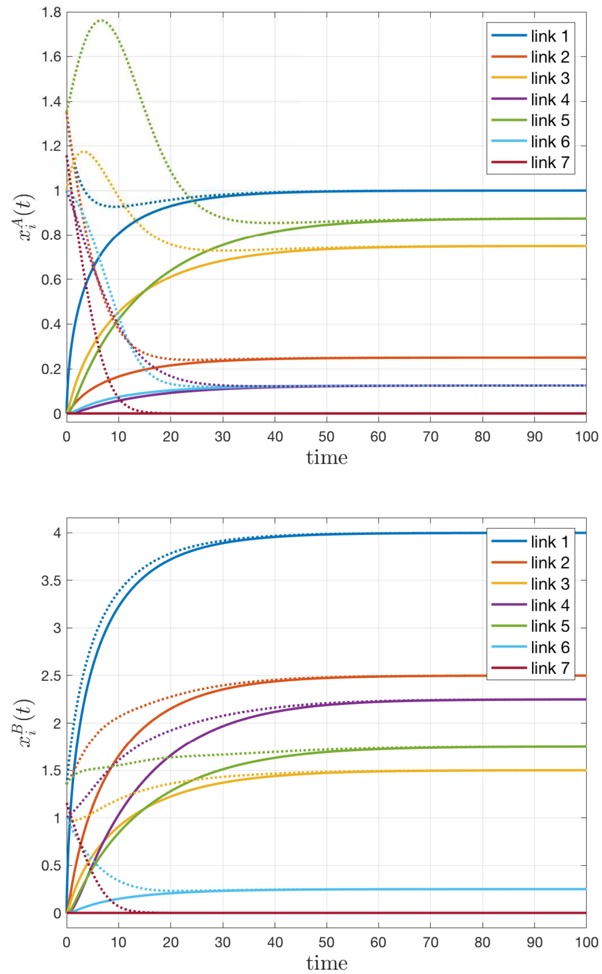

Fig. 5. The evolution of $x^A(t)$ and $x^B(t)$ for first (solild) and second (dotted) simulations

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