On Resilience of Multicommodity Dynamical Flow Networks

Gustav Nilsson, Giacomo Como, and Enrico Lovisari

***Abstract*— Dynamical flow networks with heterogeneous routing are analyzed in terms of stability and resilience to perturbations. Particles flow through the network and, at each junction, decide which downstream link to take on the basis of the local state of the network. Differently from single-commodity scenarios, particles belong to different classes, or commodities, with different origins and destinations, each reacting differently to the observed state of the network. As such, the commodities compete for the shared resource that is the flow capacity of each link of the network. This implies that, in contrast to the single-commodity case, the resulting dynamical system is not monotone, hence harder to analyze. It is shown that, in an acyclic network, when a feasible globally asymptotically stable aggregate equilibrium exists, then each commodity also admits a unique equilibrium. In addition, a sufficient condition for stability is provided. Finally, it is shown that, differently from the single-commodity case, when this condition is not satisfied, the possible unique equilibrium may be arbitrarily fragile to perturbations of the network.**

***Index Terms*— Dynamical flow networks, multicommodity flows, resilience, distributed routing, heterogeneous routing.**

I. Introduction

In a multicommodity network particles of different classes flow through a network sharing and competing for the channel resource. Examples of multicommodity flow problems are ubiquitous in engineering sciences. Traffic networks, air traffic control, data networks, production chains and supply chains can all be interpreted as multicommodity networks, where the aim is usually to let the highest possible volume of particles of the different classes through the network. Models for multicommodity flow networks based on PDEs and the celebrated LWR model have been studied [1], [2], but solutions are usually difficult to obtain even in simple settings. In this paper, we propose and analyze a dynamical ODE-based model for networks with heterogeneous routing. The network topology is modeled as a directed graph in which nodes are junctions and edges are links through which particles can flow. The flow on each link is bounded from above by a finite value called the link capacity. Particles enter in the network from origins and leave it at destinations, which possibly vary from commodity to commodity. When particles arrive at a junction, they decide which subsequent link to take on the basis of the local state of the network, that is, the aggregate of particles in each possible subsequent link. In other terms, particles are unable to distinguish in the flow the classes of particles. Routing is heterogenous in that particles of different classes have different preferred paths and react to the local state of the network in different ways. In contrast to single-commodity scenario, in which all particles belong to the same class and hence there is no competition among different classes, multicommodity networks show a complex behavior even in the static setting [3], in which it has been shown that the maximum throughput in a multicommodity network is bounded away from the value predicted by the celebrated max-flow min-cut theorem. Dynamical models based on ODEs have been proposed in the literature, but heterogeneity is usually embedded in a single-commodity scenario with fixed turning rates, i.e., in which at each junction the fraction of vehicles turning into each subsequent link is fixed [4], [5]. Differently from the latter approach, we extend the framework proposed in [6], [7] and consider a dynamic responsive scenario, in which agents have preferred paths that they would follow when completely isolated in the network, but are also willing to adapt their behavior according to the local state of the network, and avoid preferred, but highly congested, paths.

Besides the analysis of the stability of the network, we study the resilience properties of multicommodity networks with respect to perturbations, which in this paper are understood as reduction of the link's capacities. The main results of this paper are the following: 1) Under certain assumptions on the constant inflows in the network, the network admits a globally asymptotically stable equilibrium for each commodity, and 2) When the network is not single-commodity, it can be extremely fragile with respect to perturbations. In particular, perturbing a network at equilibrium can trigger a cascade effect that makes the network unstable. In addition, examples show that such a perturbation can be arbitrarily small. Such a behavior arises in multicommodities only, and has no counterpart in single-commodity network, where instead, as shown in [8], well designed routing policies can completely exploit the structure of the network and ensure maximal resilience to perturbations.

The paper is organized as follows: the rest of this section presents the notation. In Section II we provide a motiving example for the fragility of the multicommodity network. In Section III we propose a model for dynamical flow networks with heterogeneous routing. Section IV is devoted to stability analysis of the model and to a sufficient condition for the stability of the network. Section V discusses resilience and formally proves the claims made in Section II. Finally, Section VI presents some future research directions.

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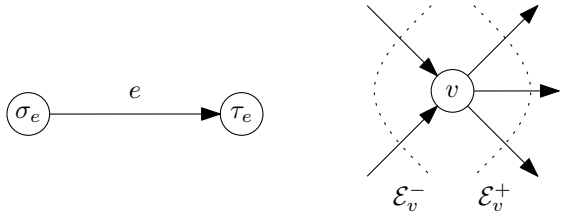
Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ denote the set of non-negative real numbers. For a set $\mathcal{A}$, $|\mathcal{A}|$ denotes its cardinality and with $\mathbb{R}^{\mathcal{A}}_{(+)}$, we mean the (non-negative) real vectors indexed by the elements in $\mathcal{A}$. In the same manner, $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}_{(+)}$ are matrices indexed by the product set of $\mathcal{A}$ and $\mathcal{B}$. A directed *multi-graph* is a pair consisting of a finite set of nodes, $\mathcal{V}$, and a finite multi-set, i.e, a set where an element is allowed to occur more than once, of directed links, $\mathcal{E}$, containing ordered pairs of nodes. For a link $e = (v_1, v_2) \in \mathcal{E}$ we write $\sigma_e = v_1$ for its tail and $\tau_e = v_2$ for its head, see Fig. 1a. The *set of outgoing links*, $\mathcal{E}^+_v$, for a node $v \in \mathcal{V}$ is defined as $\mathcal{E}^+_v := \{e \in \mathcal{E} : \sigma_e = v\}$. In the same manner the *set of incoming links* is defined as $\mathcal{E}^-_v := \{e \in \mathcal{E} : \tau_e = v\}$. The sets, for a node v, are illustrated in Fig. 1b. For sake of notation, we put $\mathcal{R} := \mathbb{R}^{\mathcal{E}}_+$.



(a) The preceding and next node for a link e

(b) The sets of incoming and outgoing edges from a node v

Fig. 1: Notation

II. A MOTIVATING EXAMPLE

Let us consider the network displayed in Fig. 2. First, we focus on single-commodity dynamical flows, where the density dynamics on each link e is described by the following conservation law

$$\dot{\rho}_e = u_e(t) - f_e(\rho_e(t)).$$

Here, $f_e(\rho_e(t))$, called the flow function of link e, represents the outflow from e, and is given by

$$f_e(\rho_e(t)) = C_e(1 - e^{-\rho_e(t)}),$$

where C_e is the link's maximum flow capacity. On the other hand, the term u_e describes how much of the flow through node σ_e should be sent to link e. In particular, we set

$$u_e(t) = G_e(\rho(t))(\lambda_{\sigma_e} + \sum_{j\in\mathcal{E}^-_{\sigma_e}} f_j(\rho_j(t))),$$

where $G_e(\rho(t))$ is a map that describes how the fraction of flow though a node that is routed towards link e depends on the current local state of the network and $\lambda_{v_1} = 2$, $\lambda_{v_i} = 0$ $i \neq 1$, denotes a static inflow at the origin node v_1.

The routing polices are constructed as

$$G_1(\rho_1,\rho_2) = \frac{e^{-\rho_1}}{e^{-\rho_1}+e^{-\rho_2}}, \quad G_3(\rho_3,\rho_4) = \frac{e^{-\rho_3}}{e^{-\rho_3}+e^{-\rho_4}},$$
$$G_2(\rho_1,\rho_2) = \frac{e^{-\rho_2}}{e^{-\rho_1}+e^{-\rho_2}}, \quad G_4(\rho_3,\rho_4) = \frac{e^{-\rho_4}}{e^{-\rho_3}+e^{-\rho_4}},$$

and $G_5(\rho_5) \equiv 1$. With these routing policies, it can be verified that the network dynamics admits an equilibrium with

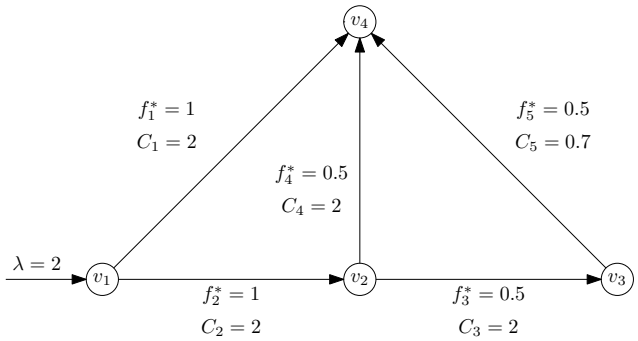


Fig. 2: A single-commodity network. The minimum residual capacity 0.2 is achieved at node v_3. Hence, under any perturbation of magnitude smaller than 0.2 the network is still be able to transfer the external inflow λ_{v_1} to the destination node v_4.

corresponding flow vector f^* whose entries are specified in Fig. 2. Such equilibrium is globally asymptotically stable [6].

We want to study how the limit flow changes when the network is perturbed, namely, when the flow capacity is reduced from C_e to $\tilde{C}_e < C_e$ on some links. Define the margin of resilience to be the infimum aggregate flow capacity reduction $\sum_{e\in\mathcal{E}}(C_e - \tilde{C}_e)$, or *perturbation magnitude*, such that the perturbed system

$$\dot{\rho}_e = \tilde{u}_e(t) - \tilde{f}_e(\rho_e(t)),$$
$$\tilde{u}_e(t) = \sum_{j\in\mathcal{E}^-_{\sigma_e}} \tilde{f}_j(\rho_j(t)) \cdot G_e(\rho(t)).$$

is unstable, i.e., the density vector $\rho(t)$ blows up in the limit of large t. For a single-commodity network, it was shown that the margin of resilience equals the minimun node residual capacity [6], [7]. This implies that the network in Fig. 2, with the given routing policies, can absorbe any perturbation of magnitude smaller than 0.2.

Now, let us move to a multicommodity scenario, where the particle density on each link is mixture of particles of two different classes, A and B, such that

$$\rho_e = \rho^A_e + \rho^B_e.$$

We assume that the particles are fully mixed, so that the dynamics for particles of class $k = A, B$ are

$$\dot{\rho}^k_e = u^k_e(t) - \frac{\rho^k_e(t)}{\rho_e(t)} f_e(\rho_e(t)),$$

where

$$u^k_e(t) = G^k_e(\rho(t))(\lambda^k_{\sigma_e} + \sum_{j\in\mathcal{E}^-_{\sigma_e}} \frac{\rho^k_j(t)}{\rho_j(t)} f_j(\rho_j(t))),$$

and $\lambda^A_{v_1} = \lambda^B_{v_1} = 1$, $\lambda^A_{v_i} = \lambda^B_{v_i} = 0$ for $i \neq 1$ are static inflows. We let the particles have different routing policies, i.e, the two commodity flows A and B have different path preferences. In particular, we consider routing polices of the

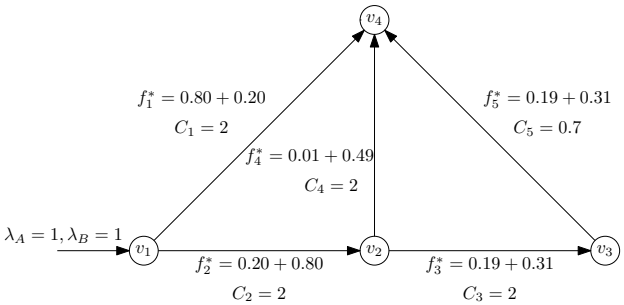
form

$$G_1^k(\rho_1,\rho_2) = 1 - G_2^k(\rho_1,\rho_2) = \frac{f_1^{k*}\cdot e^{-\alpha_1^k\rho_1}}{f_1^{k*}\cdot e^{-\alpha_1^k\rho_1} + f_2^{k*}\cdot e^{-\alpha_2^k\rho_2}}$$

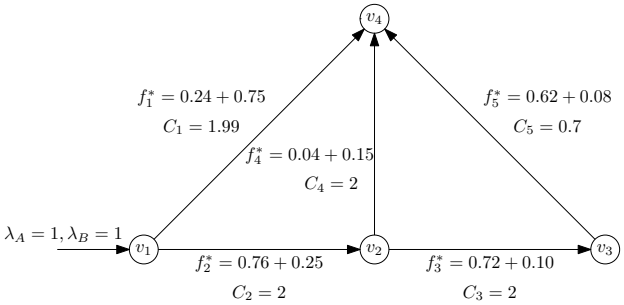
$$G_3^k(\rho_3,\rho_4) = 1 - G_4^k(\rho_3,\rho_4) = \frac{f_3^{k*}\cdot e^{-\alpha_3^k\rho_3}}{f_3^{k*}\cdot e^{-\alpha_3^k\rho_3} + f_4^{k*}\cdot e^{-\alpha_4^k\rho_4}}$$

and $G_5^A(\rho_5) \equiv G_5^B(\rho_5) \equiv 1$. Here, f_e^{k*} is the limit flow for commodity k on link e as given in Fig. 3a. Observe that the aggregate limit flows coincide with those in the single-commodity case. On the other hand, $\alpha_e^k > 0$ are parameters which do not effect the limit flows. However, these parameters do affect how the response to perturbations.

In order to illustrate the fragility of the multicommodity setting, we let $\alpha_{e_1}^A = \alpha_{e_2}^B = 1000$ and $\alpha_{e_2}^B = \alpha_{e_1}^B = 1$ and $\alpha_{e_3}^k = \alpha_{e_4}^k = 0.01$, and consider now a perturbation of magnitude 0.01 which reduces $C_{e_1} = 2$ to $\tilde{C}_{e_1} = 1.99$. The limit flows for the perturbed dynamics are shown in Fig. 3b. The perturbation causes the limit flow on link 3 to increase and exceed the capacity of the subsequent link 5. Consequently, the density on link 5 grows unbounded. This implies that the margin of resilience in the multicommodity case is not larger than 0.01. This example then indicates that a dynamical multicommodity network can be much more fragile than a single-commodity one with the same topology and aggregate equilibrium flow.



(a) Equilibrium flows for a simple network. All flows are less than edges capacities, so that the network is able to fully transfer the external inflows λ_A, λ_B to their destinations, node v_4.



(b) The same network when edge e_1's capacity is slightly decreased. Now the inflow to link e_5 is larger than its capacity, and hence the network is not able to handle the flow demands.

Fig. 3: A simple network, with two external inflows λ_A, λ_B at node v_1. Node v_4 is the destination for both flows. The flows shown in figure are given in terms of the sum of flows of the two classes on each link, i.e., $f_e^* = f_e^{A*} + f_e^{B*}$.

III. A MODEL FOR DYNAMICAL FLOW NETWORKS WITH HETEROGENOUS ROUTING

We model a dynamical multicommodity flow network as a directed multigraph $\mathcal{M} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V}$ being the set of nodes and $\mathcal{E}$ being the set of links, that is shared by a finite set $\mathcal{K}$ of different *commodities*. For every $k \in \mathcal{K}$, s_k and d_k in $\mathcal{V}$ will denote, respectively, the source and destination nodes of commodity k, and $\lambda_k \geq 0$ will stand for the inflow of such commodity at node s_k from outside the network. In order to account for the fact that, in certain applications, not all commodities can access every link, for every node v we denote by $\mathcal{E}_v^k \subseteq \mathcal{E}_v^+$ the set of *accessible (out-)links* of v for commodity k. The set of all accessible links for commodity $k \in \mathcal{K}$ is then denoted by $\mathcal{E}^k := \cup_v \mathcal{E}_v^k$. We make the following steady assumption, which ensures that particles of each commodity can reach their destination.

Assumption 1 (Existence of origin-destination paths): For $k \in \mathcal{K}$ with $\lambda_k > 0$ and every $e \in \mathcal{E}^k$ there exists a directed path in the sub-multigraph $(\mathcal{V}, \mathcal{E}^k)$ from τ_e to d_k.

Particles flow through the network queuing up on the links. We denote by $\rho_e^k \in \mathbb{R}_+$ the density of particles of commodity $k \in \mathcal{K}$ on link $e \in \mathcal{E}$, and we denote by $\rho_e := \sum_{k\in\mathcal{K}} \rho_e^k$ the aggregate density of particles on e. All particle densities in the whole network can then be described by the matrix $\rho \in \mathbb{R}_+^{\mathcal{E}\times\mathcal{K}}$. The rest of the section is devoted to describing the dynamics of the densities ρ_e^k.

To this aim, for every link e, we denote by f_e the total outflow from e, and we assume that it is a function of the aggregate density of the link, namely, $f_e = f_e(\rho_e)$. The quantity $C_e := \sup_{\rho_e\geq 0} f_e(\rho_e)$ represents the (maximum flow) *capacity* of link e. Throughout the paper, we shall refer to a *network* $\mathcal{N}$ as the pair of a topology $\mathcal{M} = (\mathcal{V}, \mathcal{E})$ and a set of flow functions $\{f_e\}_{e\in\mathcal{E}}$ satisfying the following:

Assumption 2 (Flow function): For each link $e \in \mathcal{E}$ the flow function $f_e : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing continuously differentiable function with bounded derivative, with $f_e(0) = 0$ and $C_e = \sup_{\rho_e\geq 0} f_e(\rho_e) < +\infty$.

We further make the simplifying assumption that particles of different commodities are always homogeneously mixed in each link. As a consequence, the outflow f_e^k of particles of class k from link e is proportional to the fraction of particles of class k on link e, i.e., $f_e^k = \frac{\rho_e^k}{\rho_e} f_e(\rho_e)$.

For every non-destination node $v \in \mathcal{V} \setminus \{d_k\}_{k\in\mathcal{K}}$, denote by Λ_v^k the total inflow of commodity k into v, given by

$$\Lambda_v^k := \begin{cases} \sum_{j\in\mathcal{E}_v^-} \frac{\rho_j^k(t)}{\rho_j(t)} f_j(\rho_j(t)) + \lambda_k & \text{if } v = s_k \\ \sum_{j\in\mathcal{E}_v^-} \frac{\rho_j^k(t)}{\rho_j(t)} f_j(\rho_j(t)) & \text{otherwise} \end{cases}. \quad (1)$$

For every link $e \in \mathcal{E}_v^k$, we denote by u_e^k the inflow of particles of commodity k into link e. As already mentioned, particles only queue up on links, i.e, the nodes have no buffer capacities, therefore inflows must satisfy

$$\sum_{e\in\mathcal{E}_v^k} u_e^k = \Lambda_v^k, \quad \forall v \in \mathcal{V} \setminus d_k. \quad (2)$$

Since the family of signals $\{u_e^k\}_{e\in\mathcal{E}^k,k\in\mathcal{K}}$ describes how particles split at nodes, we refer to it as a routing control.

With the previous definitions, the dynamics of density of commodity k on link e is given by the following mass-conservation law

$$\dot{\rho}_e^k = u_e^k(t) - \frac{\rho_e^k(t)}{\rho_e(t)} f_e(\rho_e(t)). \tag{3}$$

The inflow u_e^k can be in principle any signal that satisfies (2). In this paper we assume

$$u_e^k = \Lambda_v^k G_e^k(\rho). \tag{4}$$

namely, the routing control is a function of the state of the network. In particular, we consider *distributed* policies as per the following definition:

Definition 1 (Distributed routing policy): A distributed routing policy is a family of differentiable functions $\mathcal{G} := \{G_e^k : \mathcal{R} \to \mathbb{R}_+\}_{e\in\mathcal{E},k\in\mathcal{K}}$ satisfying, for all $k \in \mathcal{K}$,

a) $\sum_{e\in\mathcal{E}_v^+} G_e^k(\rho) \equiv 1$ for all $v \in \mathcal{V} \setminus \{d_k\}$

b) $G_e^k(\rho) \equiv 0$ for all $v \in \mathcal{V} \setminus \{d_k\}$, $e \notin \mathcal{E}_v^k$

c) $\dfrac{\partial G_e^k(\rho)}{\partial \rho_j} \equiv 0$ for all $v \in \mathcal{V} \setminus \{d_k\}$, $e \in \mathcal{E}_v^k$, $j \notin \mathcal{E}_v^+$

d) $\dfrac{\partial G_e^k(\rho)}{\partial \rho_j} \geq 0$ for all $v \in \mathcal{V} \setminus \{d_k\}$, $e, j \in \mathcal{E}_v^k$, $e \neq j$

e) For every $v \in \mathcal{V} \setminus \{d_k\}$ and every proper subset $\mathcal{I} \subsetneq \mathcal{E}_v^k$ there exists a continuously differentiable family of functions, $\bar{\mathcal{G}}$,

$$\bar{G}_e^k : \mathcal{R} \to \mathbb{R}_+$$

such that $\sum_{e\in\mathcal{E}_v^k} \bar{G}_e^k(\rho) \equiv 1$ and such that if

$$\rho_e \to \infty, \quad \forall e \in \mathcal{E}_v^k \setminus \mathcal{I}, \qquad \rho_j \to \rho_j^{\mathcal{I}}, \quad \forall j \in \mathcal{I},$$

then

$$\begin{aligned} G_e^k(\rho) &\to 0, && \forall e \in \mathcal{E}_v^k \setminus \mathcal{I}, \\ G_j^k(\rho) &\to \bar{G}_j^k(\rho^{\mathcal{I}}), && \forall j \in \mathcal{I}. \end{aligned}$$

We can now give the definition of a Dynamical Multicommodity Network.

Definition 2 (Dynamical multicommodity network): A dynamical multicommodity network is a network $\mathcal{N}$ associated with a family of distributed routing policies $\mathcal{G}$ and a set of commodity demands $\{s_k, d_k, \lambda_k\}_{\forall k\in\mathcal{K}}$, where the dynamics is given by (3) and controlled by (4).

Remark 1: In Definition 1, property a) ensures mass conservation at each node, while property b) ensures that particles of commodity k are routed to links on which commodity k is allowed only. For each node $v \in \mathcal{V}$ and $e \in \mathcal{E}_v^+$, property c) ensures that each $G_e^k(\rho)$ only depends on densities of links in $\mathcal{E}_v^+$, hence it is *distributed* in the sense that decisions are taken on the basis of *local information* only. Property d) describes the following *monotone* behavior: increasing the density of a link reduces the fraction of flow routed towards that link, and viceversa. Such a setting can be interpreted as an attempt to avoid congested links, and will be instrumental in the proof of our main result. Finally, property e) states that when a link is completely congested, i.e., its density is infinite, it cannot be used.

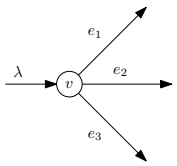


Fig. 4: A local network with $\mathcal{E}_v^+ = \{e_1, e_2, e_3\}$.

Remark 2: The definition of distributed routing policy follows the definition in [6], [7]. The key novelty is that in the present setting different commodities are allowed to have different routing policies, namely, to have different routing preferences and to respond in a different way to congestion. On the other side, all commodities compete for the same shared resource, which is the flow capacity on each of the links of the network.

Remark 3 (Loss of monotonicity): Since the controllers u_e^k and the flows are determined by the aggregate densities, the monotonicity property, that is a central property for the results in the single-commodity case, [6], [7], is no longer guaranteed.

Finally, a network that can fulfil all flow demands is called fully transferring, as per the following definition:

Definition 3 (Fully transferring): A dynamical multicommodity network is said to be fully transferring if

$$\liminf_{t\to\infty} \sum_{e\in\mathcal{E}_{d_k}^-} f_e^k(t) = \lambda_k\,, \qquad \forall\, k \in \mathcal{K}.$$

IV. Stability analysis

In this section we will state a sufficient condition for an acyclic dynamical multicommodity network to have finite limit densities and a unique limit flow. First of all, we analyze a *local network*, see Fig. 4, namely a network with a single node. For a local network, the dynamics is given by

$$\begin{aligned} \dot{\rho}_e^k =& \lambda_k(t) G_e^k(\rho(t)) \\ & - \frac{\rho_e^k(t)}{\rho_e(t)} f_e(\rho_e(t)), \quad \forall e \in \mathcal{E}_v^+, \forall k \in \mathcal{K}. \end{aligned} \tag{5}$$

From now on, we shall refer to a density ρ which is an equilibrium for (5) as a (density) equilibrium, and to the corresponding flows $\{f_e(\rho_e)\}_{e\in\mathcal{E}}$ as a flow equilibrium.

The next result offers a necessary and sufficient condition for the network to admit a globally asymptotically equilibrium.

Theorem 1: Consider a local dynamical multicommodity network $\mathcal{N}$. Assume moreover that the inflows are converging, namely $\lim_{t\to+\infty} \lambda_k(t) = \lambda_k, \forall k \in \mathcal{K}$. Then it holds that

a) if $\sum_{j\in\mathcal{J}}\lambda_j < \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e$ for every nonempty $\mathcal{J}\subseteq\mathcal{K}$, then there exists a finite unique ρ^* such that $\lim_{t\to\infty}\rho_e^k(t)=\rho_e^{k*}$ for every $e\in\mathcal{E}_v^+$ and $k\in\mathcal{K}$.
b) if there exists a nonempty $\mathcal{J}\subseteq\mathcal{K}$ such that $\sum_{j\in\mathcal{J}}\lambda_j \geq \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e$, then there exists at least one $k\in\mathcal{J}$ such that $\lim_{t\to+\infty}\rho_e(t)=+\infty$ for all $e\in\mathcal{E}_v^k$.

Theorem 1 deals with stability of a local network. In the rest of this section we shall address the stability of an acyclic network with a single origin by interpreting it as a cascade of local networks.

To this end, let $\mathcal{J}\subset\mathcal{K}$ and $\mathcal{V}^{\mathcal{J}} := \{v\in\mathcal{V}|\mathcal{E}_v^{\mathcal{J}}\neq\emptyset\}$. Moreover, let $\mathcal{U}^{\mathcal{J}}\subset\mathcal{V}^{\mathcal{J}}$ and $\partial\mathcal{U}^{\mathcal{J}} := \{e=(a,b)\in\mathcal{E}^{\mathcal{J}}|a\in\mathcal{U}, b\notin\mathcal{U}\}$. Define the minimum cut capacity between two nodes $o,s\in\mathcal{V}$, $\mathcal{C}_{o\to s}^k$ as

$$\mathcal{C}_{o\to s}^{\mathcal{J}} := \min_{\mathcal{U}^{\mathcal{J}}\subset\mathcal{V}^{\mathcal{J}}\text{ s.t }o\in\mathcal{U}^{\mathcal{J}},s\notin\mathcal{U}^{\mathcal{J}}}\sum_{e\in\mathcal{E}^+_{\partial\mathcal{U}}} C_e.$$

For sake of simplicity, consider now an acyclic network with the same origin $o\in\mathcal{V}$ for all the commodities, i.e., $s_k=o$ for all $k\in\mathcal{K}$. The following proposition offers a sufficient condition for such a network to admit unique limit flow and density.

Proposition 1: Consider an acyclic dynamical multicommodity network with single origin. Then a sufficient condition for it to admit a unique limit density and a unique limit flow is that for every $k\in\mathcal{K}$ and for every $v\in\mathcal{V}^k$

$$\min\left(\mathcal{C}_{o\to v}^{\mathcal{J}},\lambda_o^{\mathcal{J}}\right) < \sum_{e\in\mathcal{E}_v^{\mathcal{J}}} C_e. \tag{6}$$

V. Resilience

In this section we investigate how the dynamic multicommodity network responds to perturbations. In this paper a *perturbation* of a flow network corresponds to the reduction of the flow function as a function of the density, on possibly more than one link. Formally, following [6], [7], a perturbation is modeled as a family of perturbed flow functions, $\{\tilde{f}_e(\rho_e)\}_{e\in\mathcal{E}}$ such that $\tilde{f}(\rho_e)\leq f_e(\rho_e),\quad\forall e\in\mathcal{E}$ and $\tilde{f}_e$ satisfies Assumption 2. The *magnitude of the perturbation* on one link $e\in\mathcal{E}$ is then defined as $\delta_e := \sup_{\rho_e\geq 0}\tilde{f}_e(\rho_e)-f_e(\rho_e)$ and the *total magnitude* of the perturbation is then given by $\delta:=\sum_{e\in\mathcal{E}}\delta_e$. Given a family of perturbed flow functions $\{\tilde{f}_e(\rho_e)\}_{e\in\mathcal{E}}$, a perturbed network $\tilde{\mathcal{N}}$ is a network with the same graph, commodities, origin, destinations and routing policy as $\mathcal{N}$, and with flow functions $\tilde{f}$. The *resilience* of a dynamical flow network associated to a network $\mathcal{N}$ and routing policies $\mathcal{G}$ is then defined as the infimum total magnitude of perturbations making the perturbed dynamical flow network $\tilde{\mathcal{N}}$ not fully transferring.

It was proven in [6], [7] that, in the single-commodity case, the resilience of an acyclic dynamical flow network coincides with the *minimum residual capacity*, defined as

$$\min_{v\neq d}\{\sum_{e\in\mathcal{E}_v^+} C_e - f_e^*\},$$

where f^* is the limit flow of the unperturbed dynamical flow network. At the core of the result is a *diffusivity property* of single-commodity local dynamical flow networks (cf. [7, Lemma 1]) guaranteeing that a perturbation of total magnitude δ in either some of the outlinks, and/or an increase of the inflow, does not increase the limit flow of any outlink by more than the sum of δ and of the inflow increase. In other words, the network does not overreact to perturbations.

The goal of this section is to show that, when more than one commodity are present, dynamical flow networks can be instead arbitrarily fragile. In particular, we will construct a family of simple examples of multicommodity dynamical flow networks (with topology illustrated in Fig. 3) that, irrespective of their minimal residual capacity, can lose their fully transferring property even by means of arbitrarily small perturbations. This will show that their resilience equals 0.

We will proceed by first stating some properties of local multicommodity dynamical flow networks that have the fully accessible properties. The first one can be considered as a weaker version of the aforementioned diffusivity property for multicommodity dynamical networks.

Lemma 1: Consider a fully accessible local dynamical multicommodity network $\mathcal{N}$, with inflow λ such that

$$\sum_{k\in\mathcal{K}}\lambda_k < \sum_{e\in\mathcal{E}_v^+} C_e.$$

Let f^* denote the limit flow for this network. Moreover, let $\tilde{\mathcal{N}}$ be an admissible perturbed network with inflow $\tilde{\lambda}$ such that

$$\sum_{k\in\mathcal{K}}\tilde{\lambda}_k < \sum_{e\in\mathcal{E}_v^+}\tilde{C}_e.$$

Let $\tilde{f}^*(\tilde{\lambda})$ denote the limit flow of the perturbed network, with the inflows $\tilde{\lambda}$. Then for every $\mathcal{I}\subseteq\mathcal{E}_v^+$ it holds that

$$\sum_{i\in\mathcal{I}}\left(\tilde{f}_i^*(\tilde{\lambda})-f_i^*\right)\leq\sum_{k\in\mathcal{K}}\left[\tilde{\lambda}_k-\lambda_k\right]_+ + \sum_{e\in\mathcal{E}_v^+}\delta_e.$$

Lemma 1 provides a bound on the difference between aggregate limit flows before and after the perturbation in terms of its magnitude and of the difference between the inflows. Observe that, when there is only one commodity, i.e., $|\mathcal{K}|=1$, Lemma 1 reduces to Lemma 1 in [7]. On the other hand, the following two results show that, when more than one commodity is present, each commodity flow can change in an arbitrary way as long as the bound on the aggregate flow provided by Lemma 1 is satisfied.

Lemma 2: Consider a local dynamical network with two outgoing links e_1,e_2 and two commodity inflows λ_A,λ_B. Let f^{k*} be a feasible equilibrium flow. Then, for $\epsilon>0$ small enough, there exist distributed routing policies G^A and G^B such that

a) f^{k*} is the equilibrium flow of the dynamical local network,
b) there exits a perturbation of magnitude ϵ such that the perturbed limit flow, for one commodity k and for one link e, satisfies

$$\tilde{f}_e^{k*} > \min(\lambda_k, f_e^*)-\delta,$$

where $\delta > 0$ can be chosen arbitrary small.

Notice in particular that the perturbation considered in Lemma 2 does not change the inflows λ_A and λ_B, and hence by Lemma 1 $\tilde{f}_e^* \leq f_e^* + \epsilon$ for $e = e_1, e_2$. Also notice that trivially $\tilde{f}_e^{k*} \leq \min\{\lambda_k, \tilde{f}_e^*\} \leq \min\{\lambda_k, f_e^* + \epsilon\}$. Lemma 2 ensures then that after perturbation we get

$$\min(\lambda_k, f_e^*) - \delta \leq \tilde{f}_e^{k*} \leq \min(\lambda_k, f_e^* + \epsilon)\,.$$

Since ϵ and δ are arbitrary, we can steer $\tilde{f}_e^{k*}$ arbitrarily close to $\min(\lambda_k, f_e^*)$.

Lemma 3: Consider a local dynamical network, with two outgoing links e_1, e_2 and two commodity inflows λ_v^A, λ_v^B. Let f be a feasible limit flow. Then, if the commodity inflow changes to $\tilde{\lambda}$ and the new limit flows satisfy $C_{e_1} > \tilde{f}_{e_1}^* > f_{e_1}^*$ and $\tilde{f}_{e_2}^* < f_{e_2}^*$, there exist routing policies G^A, G^B, such that for a given $\delta > 0$

$$\tilde{f}_{e_1}^* > \frac{f_{e_1}^{A*}}{\lambda_A}\tilde{\lambda}_A + \frac{f_{e_1}^{B*}}{\lambda_B}\tilde{\lambda}_B - \delta.$$

We are now ready to construct an example showing that resilience can be arbitrarily low. To this aim, consider the network in Fig. 3. Start from a given feasible limit flow f^* such that

$$\gamma_1 = \frac{f_3^{A*}}{f_2^{A*}} > \gamma_2 = \frac{f_3^{B*}}{f_2^{B*}},$$

and assume that

$$\min(\lambda^A, f_2^*) > \frac{C_5 - f_2^*\gamma_2}{\gamma_1 - \gamma_2}\,.$$

We claim that we can construct routing policies such that the network will not be fully transferring after an arbitrarily small perturbation.

Consider first the local network around node v_1. Using Lemma 2, we know that can construct routing policies such that after a small perturbation on link 1 the flow of commodity A on link 2 is steered close to the value

$$\tilde{f}_{e_2}^{A*} \approx \min(\lambda_A, f_2^*) > \frac{C_5 - f_2^*\gamma_2}{\gamma_1 - \gamma_2}.$$

In node v_3, we construct then the routing policies according to Lemma 3. In this way, when, after perturbation, f_2^A approaches $\tilde{f}_2^A$ the perturbed limit flow on link 3 converges to

$$\tilde{f}_3^* = \frac{f_3^{A*}}{f_2^{A*}}\tilde{f}_2^{A*} + \frac{f_3^{B*}}{f_2^{B*}}\tilde{f}_2^{B*} > C_5.$$

Since the perturbed limit flow on link 3 is greater than the capacity of link 5, the network loses the fully transferring property, and the claim is proved.

To illustrate this behavior numerically, recall the motivating example in Section II. Since $\mathcal{C}_{v_1 \to v_3} = 2 > C_5$ the sufficient condition stated in Proposition 1 is violated. However, as the example shows, the system converges to finite limit densities. But after the perturbation on link 1, the system's perturbed limit flow $\tilde{f}_3^{A*} = 0.8 > 0.7$ and the system is not fully transferring anymore. In Fig. 5 we show how the flows on link 2 and 3 evolve, starting from zero initial state. The perturbation occurs at $t = t_p$.

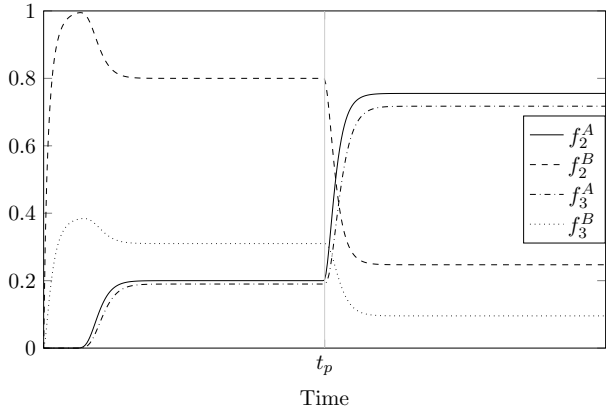


Fig. 5: The time evolution of flows on link 2 and 3. At time $t = t_p$ a perturbation occurs and causes the flow f_3 on link 3 to exceed the capacity of link 5. As a consequence, the density on link 5 grows unbounded and the network loses the property of being fully transferring.

VI. Conclusions

In this paper, a model for a dynamical multicommodity networks has been proposed and studied. A sufficient condition for the stability of the network has been provided. If the condition is violated, the network can be very fragile to perturbations, and even a small perturbation can modify the limit flows drastically and in such a way that the network becomes unstable. Future research directions include and are not limited to analysis of cyclic networks, study of the resilience under constrained routing policies, and design of robust controllers.

References

[1] J. Lebacque and M. Khoshyaran, "First order macroscopic traffic flow models for networks in the context of dynamic assignment," in *Transportation Planning*, ser. Applied Optimization, M. Patriksson and M. Labbé, Eds. Springer US, 2002, vol. 64, pp. 119–140.

[2] M. Herty, C. Kirchner, S. Moutari, and M. Rascle, "Multicommodity flows on road networks," *Communications in Mathematical Sciences*, vol. 6, no. 1, pp. 171–187, 03 2008.

[3] T. Leighton and S. Rao, "Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms," *J. ACM*, vol. 46, no. 6, pp. 787–832, Nov. 1999.

[4] P. Varaiya, "The max-pressure controller for arbitrary networks of signalized intersections," in *Advances in Dynamic Network Modeling in Complex Transportation Systems*. Springer, 2013, pp. 27–66.

[5] K. Aboudolas, M. Papageorgiou, A. Kouvelas, and E. Kosmatopoulos, "A rolling-horizon quadratic-programming approach to the signal control problem in large-scale congested urban road networks," *Transportation Research Part C: Emerging Technologies*, vol. 18, no. 5, pp. 680 – 694, 2010.

[6] G. Como, K. Savla, D. Acemoglu, M. A. Dahleh, and E. Frazzoli, "Robust distributed routing in dynamical networks — Part I: Locally responsive policies and weak resilience," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 317–332, Dec. 2013.

[7] ——, "Robust distributed routing in dynamical networks — Part II: strong resilience, equilibrium selection and cascaded failures," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 333–348, Dec. 2013.

[8] G. Como, E. Lovisari, and K. Savla, "Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing," *IEEE Transactions on Control of Network Systems*, in press, 2014.