Entropy-like Lyapunov functions for the stability analysis of adaptive traffic signal controls

Gustav Nilsson, Pouyan Hosseini, Giacomo Como, and Ketan Savla

***Abstract*— Stability of some decentralized traffic signal control policies for urban traffic networks is studied. It is proven that, whenever the arrival rates belong to a certain region — which is the largest where stability is possible— the resulting traffic network dynamics admit a globally asymptotically stable equilibrium. The results rely on the use of some entropy-like Lyapunov functions previously considered in the context of stochastic queuing networks.**

***Index Terms*— Distributed Traffic Signal Control, Nonlinear Control, Dynamical Flow Networks**

I. INTRODUCTION

Rapid advancements in traffic sensing technology have made it possible to use real-time traffic information in road traffic control. This has opened up the possibility of replacing traditional fixed-timing traffic signal controllers with adaptive controllers. Motivated by such possibilities, this paper studies stability of certain adaptive signal control policies for urban traffic networks.

References [1], [2], [3] provide an overview of the problem and practices of urban traffic signal control. Classical strategies consist of using extensive surveys to obtain network parameters, which are then used to design traffic light plans, which are either fixed, e.g., see [4], or constantly re-tuned as in SCOOT, e.g., see [5]. Classical control techniques have also been used for traffic signal control, e.g., see [6], [7]. However, these works do not provide any guarantees with respect to performance metrics of interest such as throughput, delay, and robustness to disruptions.

Recently, well-known algorithms for routing in data networks, such as the back-pressure algorithm [8] and its throughput analysis, have been adapted to the traffic signal control setting, e.g., see [9], [10], [11]. However, these algorithms require the traffic signal controllers to have explicit knowledge about the turning ratios representing the route choice behavior of drivers, a requirement that may result impractical in many real-life applications. Recently, distributed adaptive signal control algorithms that rely on the estimation of turn ratios at short time scales have also been proposed, e.g., see [11].

G. Nilsson and G. Como are with the Department of Automatic Control, Lund University, SE-221 00 Lund, Sweden {gustav.nilsson, giacomo.como}@control.lth.se.

P. Hosseini and K. Savla are with the Sonny Astani Department of Civil and Environmental Engineering, University of Southern California, Los Angeles, CA , USA, 90089-2531, {seyedpoh, ksavla}@usc.edu.

G. Nilsson and G. Como are members of the excellence centers ELLIT and LCCC and were supported by the Swedish Research Council through the Junior Research Grant 'Information Dynamics in Large Scale Networks'. P. Hosseini and K. Savla were partially supported by METRANS Research Initiation Award Grant No: 65A0533 from Caltrans. K. Savla was also supported by NSF CAREER ECCS Project No: 1454729.

In this paper, we consider adaptive traffic signal controls of the form first introduced in our previous works [12], [13]. These controls are completely decentralized, in that the traffic signals at an intersection are chosen as a function of the densities in the incoming lanes to that intersection only, and universal, in that they do not require any knowledge of either the capacity of the lanes, the turning ratios, or the arrival rates. Our main contribution is proving that these policies are maximally stabilizing in the sense that, for the largest possible arrival rate region, they drive the traffic network dynamics to globally asymptotically stable equilibria.

The main novelty of this paper with respect to [12], [13] is the use of entropy-like Lyapunov functions in the stability analysis, as opposed to monotonicity and l_1-contraction arguments employed there. This different approach proves stronger in that it allows us to deal with cyclic networks (while in [12], [13] we were restricted to acyclic networks).

Our use of entropy-like Lyapunov functions is inspired by results in packet-switched stochastic queuing networks, see e.g., the recent work [14]. In particular, some of the technical results in the proofs rely on adaptations of arguments developed in the context of proportional fairness bandwidth allocation problems [15]. The use of these techniques in the context of traffic signal control in urban networks is a novel contribution to our knowledge.

The rest of this paper is organized as follows. In Section II we introduce the dynamical model of urban traffic network with signalized intersections, and describe a class of decentralized universal green-line policies. In Section III we prove that, in the special case when only single lanes can be activated at every intersection, such green light policies drive the system to a globally asymptotically stable equilibrium. In Section IV the proposed green light policy from Section III is simulated and also a simulation for when multiple lanes can be activated simultaneously is shown. The paper is concluded by an Appendix containing a few key technical lemmas.

A. Notation

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+$ the set of nonnegative reals. For finite sets $\mathcal{A}$ and $\mathcal{B}$, let $|\mathcal{A}|$ denote the cardinality of $\mathcal{A}$ and $\mathbb{R}^{\mathcal{A}}$ the space of real-valued vectors whose elements are indexed by $\mathcal{A}$. For a set of vectors $\mathcal{K} \subseteq \mathbb{R}^{\mathcal{A}}$, $\mathrm{conv}(\mathcal{K})$ will stand for its convex hull.

Let $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ denote a directed graph where $\mathcal{E}$ is the set of directed links and $\mathcal{V}$ is the set of vertices or nodes. For each link $e = (i, j) \in \mathcal{E}$, let $\tau_e = j \in \mathcal{V}$ denote the head of the link e and $\sigma_e = i \in \mathcal{V}$ the tail of the link e.

For each node $v \in \mathcal{V}$, introduce the set of incoming links as $\mathcal{E}_v := \{e \in \mathcal{E} : \tau_e = v\}$.

II. MODEL

We describe the topology of a urban traffic network as a capacitated directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$, whose nodes $v \in \mathcal{V}$ represent junctions and whose links $i \in \mathcal{E}$ represent lanes, and where $C \in \mathbb{R}^{\mathcal{E}}$ is a vector whose entries $C_i > 0$ represent the flow capacities of the lanes $i \in \mathcal{E}$. Traffic flows among consecutive lanes according to a *routing matrix* $R \in \mathbb{R}_+^{\mathcal{E}\times\mathcal{E}}$ whose (i,j)-th entry R_{ij} —which will be referred to as a *turning ratio*— represents the fraction of flow out of lane i that joins lane j. Conservation of mass implies that $\sum_{j\in\mathcal{E}} R_{ij} \leq 1$ for all $i \in \mathcal{E}$, the quantity $1 - \sum_{j\in\mathcal{E}} R_{ij} \geq 0$ representing the fraction of flow out of lane i that leaves the network directly. In other words, the routing matrix R is sub-stochastic. Moreover, the natural topological constraints encoded in the graph $\mathcal{G}$ imply that $R_{ij} = 0$ if $\tau_i \neq \sigma_j$, i.e., $R_{ij} = 0$ whenever lane i does not end in the junction where lane j starts. We will refer to this property of the routing matrix R as being *adapted* to $\mathcal{G}$. Finally, we consider an *arrival vector* $\lambda \in \mathbb{R}_+^{\mathcal{E}}$, whose entries $\lambda_i \geq 0$ describe the external inflows on the lanes $i \in \mathcal{E}$.

Throughout the paper, we will make the following assumption on the network topology $\mathcal{G}$, the routing matrix R, and the arrival vector λ.

Assumption 1: The routing matrix R is adapted to the topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$. Moreover, for every road $i \in \mathcal{E}$,

(i) there exists some $k \in \mathcal{E}$ such that $\sum_{j\in\mathcal{E}} R_{kj} < 1$ and $(R^l)_{ik} > 0$ for some $l \geq 0$;

(ii) there exists some $h \in \mathcal{E}$ such that $\lambda_h > 0$ and $(R^l)_{hi} > 0$ for some $l \geq 0$.

Part (i) of Assumption 1 states that from every lane $i \in \mathcal{E}$ it is possible to reach some lane $k \in \mathcal{E}$ with $\sum_{j\in\mathcal{E}} R_{kj} < 1$ by a length-l path $i = j_0, j_1, \ldots, j_l = k$ such that $R_{j_{s-1},j_s} > 0$ for all $1 \leq s \leq l$. Physically, this ensures that from every lane there exists a path to an exit of the network. Mathematically, this implies that the spectral radius of R (hence, of its transpose R^T) is strictly smaller than 1: in particular, this ensures that the matrix $(I - R^T)$ is invertible with inverse nonnegative inverse $(I - R^T)^{-1} = \sum_{k\geq 0}(R^T)^k$. On the other hand, part (ii) of Assumption 1 states that every lane $i \in \mathcal{E}$ is reachable by some lane $h \in \mathcal{E}$ with positive external inflow $\lambda_h > 0$ by a length-l path $h = j_0, j_1, \ldots, j_l = i$ such that $R_{j_{s-1},j_s} > 0$ for all $1 \leq s \leq l$. In particular, this implies that all the entries of the vector $(I - R^T)^{-1}\lambda = \sum_{k\geq 0}(R^T)^k\lambda$ are strictly positive.

In order to complete the description of the urban traffic network, we need to introduce the notion of *phases*. These are subsets of lanes that can be given green light simultaneously. We will thus identify every phase with a binary vector $p \in \{0,1\}^{\mathcal{E}}$ whose i-th entry p_i equals 1 if lane i receives green light during phase p and 0 otherwise. The set of all possible phases will be denoted by $\Psi \subseteq \{0,1\}^{\mathcal{E}}$.

We will then study continuous-time dynamics with state vector $\rho(t) \in \mathbb{R}_+^{\mathcal{E}}$ whose entries $\rho_i(t)$ denote the traffic volume on the lanes $i \in \mathcal{E}$. Such dynamics are of the form

$$\dot{\rho}_i = \lambda_i + \sum_{j\in\mathcal{E}} R_{ji} C_j h_j(\rho) - C_i h_i(\rho), \quad \forall i \in \mathcal{E}\,. \qquad (1)$$

In equation (1) above, when $\rho_i > 0$, the term $h_i(\rho)$ represents the total fraction of time that lane i is given green light. This can be expressed as

$$h_i(\rho) = \sum_{p\in\Psi} \theta_p(\rho) p_i\,, \qquad \text{if } \rho_i > 0\,, \qquad (2)$$

where $\theta_p(\rho)$ represents the fraction of time that phase p is activated. Here, $\theta(\rho)$ is a *green light (feedback) policy*: the domain of θ is $\mathbb{R}_+^{\mathcal{E}}$, while its range is the simplex $\mathcal{S}$ of probability vectors over the set of phases Ψ. In other terms, for all network states $\rho \in \mathbb{R}_+^{\mathcal{E}}$, $\theta(\rho)$ is a vector with nonnegative entries $\theta_p(\rho)$ indexed by the phases $p \in \Psi$, such that $\sum_{p\in\Psi} \theta_p(\rho) = 1$.

Remark 1: Often traffic light policies are designed in a discrete time setting. The continuous green light policies in this paper can be interpreted as the time-averaged green light duration. However, the rigorous investigation of how to connect the continuous green light policies to a discrete time setting is a topic left for future research.

Remark 2: Equation (2) characterizes the value of $h_i(\rho)$ only when $\rho_i > 0$. The case when $\rho_i = 0$ has to be treated specifically in order to guarantee the physically obvious requirement that the dynamical system (1) keeps the nonnegative orthant $\mathbb{R}_+^{\mathcal{E}}$ invariant. In the special case when there are only single phases, this issue is easily dealt with for the specific green light policies considered in this paper, as shown in Section III. For multiphases, the issue is more delicate, as discussed more in detail in Section IV, and will be dealt with in another publication.

Observe that if an equilibrium ρ^* of the dynamical system (1) exists with all positive entries, it must satisfy

$$0 = \lambda_i + \sum_{j\in\mathcal{E}} R_{ji} C_j h_j(\rho^*) - C_i h_i(\rho^*),$$

which can be compactly written as

$$\lambda + (R^T - I)\mathrm{diag}\,(C)\, h(\rho^*) = 0,$$

or

$$h(\rho^*) = \mathrm{diag}\,(C)^{-1}\, a, \qquad a := (I - R^T)^{-1}\lambda\,. \qquad (3)$$

This argument implies the following result.

Proposition 1 (Necessary condition for stability): Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a traffic network topology, R a routing matrix adapted to $\mathcal{G}$, and $\lambda \in \mathbb{R}_+^{\mathcal{E}}$ an arrival vector such that Assumption 1 is satisfied. Let $\Psi \subseteq \{0,1\}^{\mathcal{E}}$ be a nonempty set of phases, $\mathcal{S}$ the simplex of probability vectors over Ψ, and $\theta : \mathbb{R}_+^{\mathcal{E}} \to \mathcal{S}$ a green light policy. If the dynamical system (1) admits an equilibrium ρ^* with all positive entries, then it must hold that

$$\mathrm{diag}\,(C)^{-1}\, a \in \mathrm{conv}(\Psi)\,, \qquad (4)$$

where

$$a = (I - R^T)^{-1}\lambda\,. \qquad (5)$$

Proof: If ρ^* is an equilibrium of (1) whose entries are all positive, then it follows from (2) and (3) that

$$\operatorname{diag}(C)^{-1} a = \sum_{p\in\Psi} \theta_p(\rho) p\,,$$

i.e., $\operatorname{diag}(C)^{-1} a$ is a convex combination of phases. ■

In this paper we will focus on the case where $\operatorname{diag}(C)^{-1} a \in \operatorname{int}(\operatorname{conv}(\Psi))$ and study green light policies that admit (globally asymptotically) stable equilibria. We will consider sets of phases that model local constraints among the incoming lanes in each intersection $v \in \mathcal{V}$. Specifically, observe that the set of lanes can be partitioned as $\mathcal{E} = \cup_{v\in\mathcal{V}}\mathcal{E}_v$, where $\mathcal{E}_v$ stands for the set of lanes incoming junction v. We will then assume that:

Assumption 2: The set of phases $\Psi = \prod_{v\in\mathcal{V}} \Psi_v$, where $\Psi_v \subseteq \{0,1\}^{\mathcal{E}_v}$ is the local set of phases at junction $v \in \mathcal{V}$. Moreover, each local of set of phases Ψ_v contains the all-zero phase $0 \in \Psi_v$.

The Assumption 2 ensures that there are no joint constraints among the green lights that can be activated simultaneously at the different intersections.

We will then focus on green light policies $\theta(\rho)$ that can be written as the concatenation of local policies $\theta^{(v)}(\rho^{(v)})$ —where $\rho^{(v)} = (\rho_i)_{i\in\mathcal{E}_v}$ is the vector of densities on the lanes incoming junction $v \in \mathcal{V}$— of the following form

$$\theta^{(v)}(\rho^{(v)}) \in \underset{\theta\in\mathcal{S}_v}{\operatorname{argmax}} \sum_{i\in\mathcal{E}_v} \rho_i \log(\sum_{p\in\Psi_v} \theta_p p_i) + \kappa_v \log\theta_0\,, \quad (6)$$

where $\mathcal{S}_v$ is the simplex of probability vectors over Ψ_v and $\kappa_v > 0$ is the *zero phase weight*. The zero phase is introduced to capture the fact that under normal traffic demands, a fraction of the possible utilization is used to phase shifts.

From now on, with a slight abuse of notation, we will refer to

$$h^{(v)}(\rho^{(v)}) = \sum_{p\in\Psi_v} \theta_p^{(v)}(\rho^{(v)}) p\,, \qquad \forall v \in \mathcal{V} \qquad (7)$$

as the *maximizing green light policy*.

III. Stability analysis in the single phase case

In this section we focus on the special case of phase sets that do not allow for multiphases, i.e., where every phase can prescribe green light to at most one lane incoming to a junction. Specifically, we assume that the local set of phases at every intersection is

$$\Psi_v = \{p \in \{0,1\}^{\mathcal{E}_v} : \sum_{e\in\mathcal{E}_v} p_e \le 1\}\,, \qquad \forall v \in \mathcal{V}. \qquad (8)$$

In this case, the necessary condition for stability (4) takes the form

$$a_i \ge 0\,, \quad \forall i \in \mathcal{E}\,, \qquad \sum_{i\in\mathcal{E}_v} \frac{a_i}{C_i} < 1\,, \qquad \forall v \in \mathcal{V}\,.$$

Moreover, the green light policy can be expressed explicitly as in the following result.

Lemma 1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a traffic network topology and $\Psi = \prod_{v\in\mathcal{V}} \Psi_v$ a set of phases satisfying (8). Then, for every junction $v \in \mathcal{V}$, and every strictly positive local state vector $\rho^{(v)}$, the maximizing green light policy satisfies

$$h_i^{(v)}(\rho^{(v)}) = \frac{\rho_i}{\sum_{j\in\mathcal{E}_v} \rho_j + \kappa_v}, \quad \forall i \in \mathcal{E}_v\,.$$

Using the expression above, $h^{(v)}$ can be extended by continuity to $\rho^{(v)} \in \mathbb{R}_+^{\mathcal{E}_v}$.

Proof: Let us identify the set of lanes $\mathcal{E}_v$ into junction v with the integers $1, \dots, k$, where $k := |\mathcal{E}_v|$. Then, for the set of phases given by (8), the maximization problem in (6) reduces to

$$\begin{aligned} &\text{maximize} \sum_{1\le i\le k} \rho_i \log(\theta_i) + \kappa_v \log(\theta_0) \\ &\text{subject to} \sum_{0\le i\le k} \theta_i = 1\,, \text{ and } \theta_i \ge 0 \text{ for } 0 \le i \le k. \end{aligned}$$

Let γ be the Lagrange multiplier associated to the equality constraint. The Lagrangian of the relaxed problem without nonnegativity constraints is then

$$f(\theta,\gamma) = \sum_{1\le i\le k} \rho_i \log(\theta_i) + \kappa_v \log(\theta_0) + \gamma\left(\sum_{0\le i\le k} \theta_i - 1\right).$$

The zero gradient conditions

$$\begin{aligned} \frac{\partial f}{\partial\theta_i} &= \frac{\rho_i}{\theta_i} + \gamma = 0, \quad \forall 1 \le i \le k, \\ \frac{\partial f}{\partial\theta_0} &= \frac{\kappa_v}{\theta_0} + \gamma = 0, \\ \frac{\partial f}{\partial\gamma} &= \sum_{0\le i\le k} \theta_i - 1 = 0, \end{aligned}$$

then give that

$$\theta_i = -\frac{\rho_i}{\gamma} = \frac{\rho_i}{\sum_{j\in\mathcal{E}_v} \rho_j + \kappa_v} \ge 0, \quad \forall 1 \le i \le k.$$

Since the objective function is concave, this is the maximizing green light policy. ■

Using the explicit expression above allows one to prove the following stability result.

Theorem 1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a traffic network topology, R a routing matrix, and $\lambda \in \mathbb{R}_+^{\mathcal{E}}$ an arrival vector such that Assumption 1 is satisfied. Let $\Psi = \prod_{v\in\mathcal{V}} \Psi_v$. Then the dynamical system (1), with green light policies given by (7), satisfying

$$\sum_{i\in\mathcal{E}_v} \frac{a_i}{C_i} < 1, \quad \forall v \in \mathcal{V}, \qquad (9)$$

admits a globally asymptotically stable equilibrium ρ^*, where

$$\rho^{(v)*} = \kappa_v \left(I - \left(\frac{a_i}{C_i}\right)_{i\in\mathcal{E}_v} \mathbf{1}^T\right)^{-1} \left(\frac{a_i}{C_i}\right)_{i\in\mathcal{E}_v}$$

for all $v \in \mathcal{V}$.

Proof: Using the explicit expression for the maximizing green light policy given in Lemma 1, together with the stability condition (9) for invertiablity of $(I - (\frac{a_i}{C_i})_{i \in \mathcal{E}_v} \mathbf{1}^T)$, yields the expression for the limit densities.

To prove that ρ^* is globally asymptotically stable, first observe that $\rho(t) \geq 0$ for all $t \geq 0$, since when $\rho_i = 0$ for $i \in \mathcal{E}$, $h_i(\rho) = 0$ and $\dot{\rho}_i \geq 0$.

Introduce the function $V : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}$ as

$$V(\rho) = \sum_{i \in \mathcal{E}} \rho_i \log\left(\frac{C_i h_i(\rho)}{a_i}\right) + \sum_{v \in \mathcal{V}} \kappa_v \log\left(\frac{h_0^{(v)}(\rho)}{h_0^{(v)}(\rho^*)}\right), \tag{10}$$

where, with a slight abuse of notation,

$$h_0^{(v)}(\rho) = 1 - \sum_{i \in \mathcal{E}_v} h_i(\rho).$$

We will now prove that $V(\rho)$ is a Lyapunov function.

Negative drift $\dot{V}(\rho) < 0$ for all $\rho \neq \rho^*$: By use of Lemma 2, it holds that

$$\frac{dV}{dt} = \sum_{i \in \mathcal{E}} \frac{\partial V}{\partial \rho_i} \frac{d\rho_i}{dt}$$
$$= \sum_{i \in \mathcal{E}} \Big(\sum_{j \in \mathcal{E}} R_{ji} h_j(\rho) C_j - h_i(\rho) C_i + \lambda_i \Big) \log\left(\frac{h_i(\rho) C_i}{a_i}\right).$$

Introduce $u_i = \log\left(\frac{h_i(\rho)C_i}{a_i}\right)$, then the time derivative can be written as

$$\frac{dV}{dt} = \sum_{i \in \mathcal{E}} \frac{\partial V}{\partial \rho_i} \frac{d\rho_i}{dt} = \sum_{i \in \mathcal{E}} \left(\sum_{j \in \mathcal{E}} R_{ji} a_j e^{u_j} - a_i e^{u_i} + \lambda_i \right) u_i,$$

or equivalently in matrix form as

$$\frac{dV}{dt} = u^T(\lambda - (I - R^T)(\operatorname{diag}(a)\, e^u)).$$

From here, Lemma 3 ensures that the Lyapunov function has negative drift for all $u \neq 0$, when $u = 0$, it holds that $h_i(\rho) = \frac{a_i}{C_i}$ which is the equilibrium.

$V(\rho^*) = 0$ and $V(\rho) > 0$: Since $h_i(\rho^*) = \frac{a_i}{C_i}$ for all $i \in \mathcal{E}$ at equilibrium, it follows that $V(\rho^*) = 0$. Moreover, if $\rho \neq \rho^*$, then

$$V(\rho) =$$
$$\sum_{i \in \mathcal{E}} \rho_i \log\left(\frac{C_i h_i(\rho)}{a_i}\right) + \sum_{v \in \mathcal{V}} \kappa_v \log\left(\frac{h_0^v(\rho)}{h_0^v(\rho^*)}\right)$$
$$> \sum_{i \in \mathcal{E}} \rho_i \log\left(\frac{C_i h_i(\rho^*)}{a_i}\right) + \sum_{v \in \mathcal{V}} \kappa_v \log\left(\frac{h_0^v(\rho^*)}{h_0^v(\rho^*)}\right) = 0,$$

where the strict inequality follows from the suboptimality of the strictly concave optimization problem for $\rho > 0$. If $\rho_i = 0$ for a subset $\tilde{\mathcal{E}} \subseteq \mathcal{E}$, the above inequality is still strict, due to the fact that for every node $v \in \mathcal{V}$ it holds that

$$\sum_{i \in \mathcal{E}_v \setminus \tilde{\mathcal{E}}_v} h_i(\rho^*) + h_0^v(\rho^*) < \sum_{i \in \mathcal{E}_v \setminus \tilde{\mathcal{E}}_v} h_i(\rho) + h_0^v(\rho).$$

$V(\rho)$ radially unbounded: Due to the stability condition (9), it is possible to choose an $\epsilon > 0$, such that $\tilde{h}_i = \frac{a_i}{C_i} + \epsilon$ and $\tilde{h}_0^{(v)} = 1 - \sum_{i \in \mathcal{E}_v} \frac{a_i}{C_i} - n\epsilon$ where $n = |\mathcal{E}_v|$. $\tilde{h}^{(v)}$ is a feasible but not optimal solution to the maximization problem stated in (7). Hence, due to suboptimality it holds that

$$V(\rho) >$$
$$\sum_{i \in \mathcal{E}} \rho_i \log\left(1 + \epsilon \frac{C_i}{a_i}\right) + \sum_{v \in \mathcal{V}} \kappa_n \log\left(\frac{\tilde{h}_0^{(v)}}{h_0^{(v)}(\rho^*)}\right) \to \infty,$$

when $|\rho| \to \infty$.

■

IV. Simulation results

The green light policy proposed in this paper is implemented with $\kappa = 2.5$ for a network of four intersections, as shown in Fig. 1. Each intersection has 12 incoming lanes. Turn ratios are assumed to be 0.17 for left turning, 0.33 for through movement and 0.5 for right turning. The lane flow capacities are symmetrical throughout the network and are specified to be 1.5 for the left lane, 1.6 for the middle lane and 1.7 for the right lane. The evolution of lane occupancies is presented in Fig. 3.

In the first simulation we use the single phase policy proposed in Section III. External arrival rate λ_i is considered to be 0.35 for all lanes located on external roads of the network. We assume that the arrival rate is zero for internal lanes of network. How the dynamics evolves is shown in Fig. 3.

To extended our example with multiphases, we introduce four phases as illustrated in Fig. 2. The set of phases, Ψ_v, then contains the vectors

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T,$$
$$\begin{bmatrix}0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\end{bmatrix}^T,$$
$$\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^T,$$
$$\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}^T,$$

to model the four phases p_1, p_2, p_3 and p_4 respectively. Moreover, Ψ_v is constructed such that if $p \in \Psi_v$, then $q \in \Psi_v$ for all $q \in \{0,1\}^{\mathcal{E}_v}$ such that $q \leq p$, which then always enables us to give zero green light to an empty lane in the simulations without disabling the whole phase. In this case larger external arrival rate can be used, and we let $\lambda_i = 1$. The dynamics for the multiphase case is shown in Fig. 4.

The simulations show that even if phases that allows several lanes to activated at the same time are introduced, the dynamical system seems to remain stable. An argument along the lines of the one developed in the single phase case allows one to prove the existence of a locally stable equilibrium. However, further theoretical work is needed to work out how different phase constructions actually effects the existence and uniqueness of equilibrium in the multi-phase case. For instance, the simulation shows that some of the lanes will have zero occupancy when multiphases are allowed. Another issue is to find an efficient numerical solver for the maximization problem.

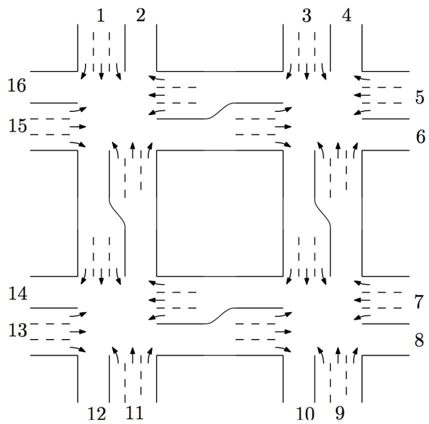

Fig. 1: Network of four intersections used in the simulations

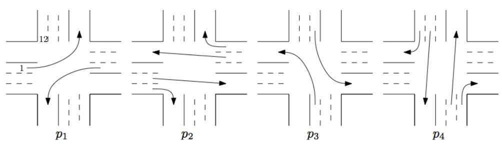

Fig. 2: Illustration of a four phase architecture at an intersection, where the incoming lanes are numbered counterclockwise from 1 to 12

V. Conclusion

We studied stability of some decentralized traffic signal control policies for urban traffic networks. Our main theoretical result shows that, in the case when only single phases are allowed, the resulting traffic network dynamics admit a globally asymptotically stable equilibrium, provided that the arrival rates belong to the interior of a certain stability polytope. These results rely on the use some entropy-like Lyapunov functions previously considered in the context of stochastic queuing networks.

Future works should consider a theoretical investigation of multiphases, dynamic route choice behavior, finite capacities on lane occupancies, and designing phases with respect to different performance measurements such as throughput and delay.

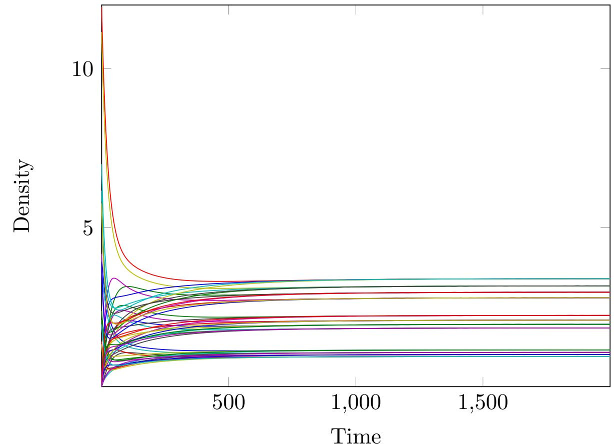

Fig. 3: Evolution of lane occupancies under proposed green light policy with only single phases

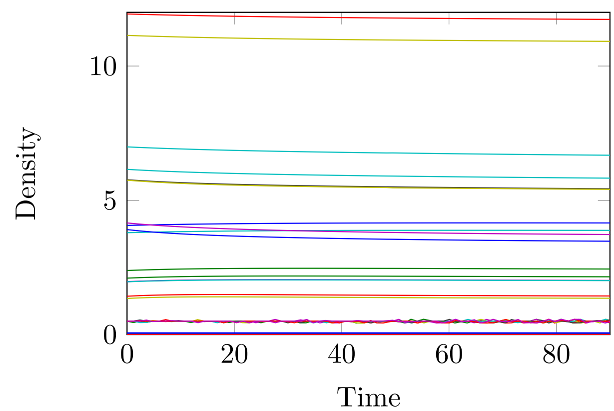

Fig. 4: Evolution of lane occupancies under proposed green light policy with multiphases

References

[1] M. Papageorgiou, C. Diakaki, V. Dinopoulou, A. Kotsialos, and Y. Wang, "Review of road traffic control strategies," *Proceedings of the IEEE*, vol. 91, no. 12, pp. 2043–2067, 2003.

[2] M. Papageorgiou, M. Ben-Akiva, J. Bottom, P. Bovy, S. Hoogendoorn, N. Hounsell, A. Kotsialos, and M. McDonald, "Chapter 11 ITS and traffic management," in *Transportation*, ser. Handbooks in Operations Research and Management Science, C. Barnhart and G. Laporte, Eds. Elsevier, 2007, vol. 14, pp. 715 – 774.

[3] A. Stevanovic, *Adaptive traffic control systems: domestic and foreign state of practice*, 2010, no. Project 20-5 (Topic 40-03).

[4] D. Robertson, "The TRANSYT method of coordinating traffic signals," *Traffic Engineering & Control*, vol. 38, no. 2, pp. 76–77, 1997.

[5] R. Bretherton, K. Wood, and G. Bowen, "SCOOT version 4," in *Proceedings of 9th International Conference on Road Transport Information and Control*, 1998.

[6] C. Cai, C. Wong, and B. Heydecker, "Adaptive traffic signal control using approximate dynamic programming," *Transportation Research Part C: Emerging Technologies*, vol. 17, no. 5, pp. 456 – 474, 2009.

[7] K. Aboudolas, M. Papageorgiou, A. Kouvelas, and E. Kosmatopoulos, "A rolling-horizon quadratic-programming approach to the signal control problem in large-scale congested urban road networks," *Transportation Research Part C: Emerging Technologies*, vol. 18, no. 5, pp. 680 – 694, 2010.

[8] L. Tassiulas and A. Ephremides, "Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks," *Automatic Control, IEEE Transactions on*, vol. 37, no. 12, pp. 1936–1948, 1992.

[9] T. Wongpiromsarn, T. Uthaicharoenpong, Y. Wang, E. Frazzoli, and D. Wang, "Distributed traffic signal control for maximum network throughput," in *Intelligent Transportation Systems (ITSC), 2012 15th International IEEE Conference on*. IEEE, 2012, pp. 588–595.

[10] P. Varaiya, "The max-pressure controller for arbitrary networks of signalized intersections," in *Advances in Dynamic Network Modeling in Complex Transportation Systems*. Springer, 2013, pp. 27–66.

[11] T. Le, P. Kovács, N. Walton, H. L. Vu, L. L. Andrew, and S. S. Hoogendoorn, "Decentralized signal control for urban road networks," *Transportation Research Part C: Emerging Technologies*, 2015.

[12] K. Savla, E. Lovisari, and G. Como, "On maximally stabilizing adaptive traffic signal control." in *Allerton*, 2013, pp. 464–471.

[13] ——, "On maximally stabilizing adaptive signal control for urban traffic networks under multi-movement phase architecture," in *19th IFAC World Congress*, 2014.

[14] N. S. Walton, "Concave switching in single and multihop networks," *SIGMETRICS Perform. Eval. Rev.*, vol. 42, no. 1, pp. 139–151, 2014.

[15] L. Massoulié, "Structural properties of proportional fairness: stability and insensitivity," *The Annals of Applied Probability*, pp. 809–839, 2007.

Appendix

Additional Lemmas

Lemma 2: Let $V(\rho)$ be as in (10), then for all $\rho > 0$,

$$\frac{\partial V(\rho)}{\partial \rho_i} = \log\left(\frac{C_i h_i(\rho)}{a_i}\right).$$

Proof: Let ρ^ξ be a vector, such that $\rho_i^\xi = \rho_i + \xi$ and $\rho_j^\xi = \rho_j$, $j \neq i$. Then

$$\begin{aligned}
&V(\rho^\xi) - V(\rho) = \\
&\sum_{i\in\mathcal{E}} \rho_i^\xi \log\left(\frac{C_i h_i(\rho^\xi)}{a_i}\right) + \sum_{v\in\mathcal{V}} \kappa_v \log\left(\frac{h_0^{(v)}(\rho^\xi)}{h_0^{(v)}(\rho^*)}\right) \\
&- \left(\sum_{i\in\mathcal{E}} \rho_i \log\left(\frac{C_i h_i(\rho)}{a_i}\right) + \sum_{v\in\mathcal{V}} \kappa_v \log\left(\frac{h_0^{(v)}(\rho)}{h_0^{(v)}(\rho^*)}\right)\right) \\
&\geq \sum_{i\in\mathcal{E}} \rho_i^\xi \log\left(\frac{C_i h_i(\rho)}{a_i}\right) + \sum_{v\in\mathcal{V}} \kappa_v \log\left(\frac{h_0^{(v)}(\rho)}{h_0^{(v)}(\rho^*)}\right) \\
&- \left(\sum_{i\in\mathcal{E}} \rho_i \log\left(\frac{C_i h_i(\rho)}{a_i}\right) + \sum_{v\in\mathcal{V}} \kappa_v \log\left(\frac{h_0^{(v)}(\rho)}{h_0^{(v)}(\rho^*)}\right)\right) \\
&= \xi \log\left(\frac{C_i h_i(\rho)}{a_i}\right),
\end{aligned}$$

where the inequality follows from the suboptimality of the solution. In the same manner, we have that

$$V(\rho^\xi) - V(\rho) \leq \xi \log\left(\frac{C_i h_i(\rho^\xi)}{a_i}\right).$$

The combination of the two inequalities yields

$$\log\left(\frac{C_i h_i(\rho)}{a_i}\right) \leq \frac{1}{\xi}\left(V(\rho^\xi) - V(\rho)\right) \leq \log\left(\frac{C_i h_i(\rho^\xi)}{a_i}\right).$$

Letting $\xi \to 0$ proves the lemma. ■

Lemma 3: Let R be a routing matrix and λ an arrival vector satisfying Assumption 1. Then, for $a \in \mathbb{R}_+^{\mathcal{E}}$ satisfying the relation,

$$a = (I - R^T)^{-1}\lambda,$$

and for any $u \in \mathbb{R}_+^{\mathcal{E}}$ it holds that

$$u^T(\lambda - (I - R^T)(\mathrm{diag}(a)\, e^u)) \leq 0,$$

with equality if and only if $u = 0$.

Proof: See [15, Lemma 7]. ■