On Robustness of Equilibria in Dynamical Transportation Networks

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***Abstract*— With growing traffic demands, transportation networks become more and more congested and prone to disruptions. In this paper, we study how different perturbations affect the free flow equilibria, i.e., equilibria where no congestion effects are present, in transportation networks. More specifically, if and when the equilibrium of the perturbed dynamics is still in free flow. A generalized cell transmission model models the dynamics of the transportation network, and the perturbations considered are perturbations in the exogenous inflows, flow capacity drops, and perturbations in the routing, i.e., when drivers deviate from their normal route preferences. The paper also shows that social optimum assignments in traffic networks, may not be the most robust ones.**

I. Introduction

Ever growing traffic loads, limited infrastructure capacity, and fast increasing interconnectedness and complexity are making the transportation system more and more sensitive to disturbances and prone to disruptions. Being able to quantify robustness of transportation networks is of importance as many of the control actions rely on estimation techniques whose uncertainty should be taken into account when, e.g., designing adaptive traffic signal controls and variable speed limits or giving route recommendations.

This paper is concerned with the robustness analysis of dynamical traffic flows in transportation networks. We model the network traffic flow dynamics using a generalized continuous time version of the cell transmission model (CTM) originally presented in [1], [2] as a discretization and network generalization of the Lighthill-Whitham-Richards traffic flow model [3], [4]. We study robustness of so-called free-flow equilibrium points in such models with respect to perturbations in the exogenous inflows, the cell capacities, and the routing matrix. To such perturbations we assign a cost function and then quantify the transportation network's robustness as the minimum cost of a perturbation such that the network does not admit a free-flow equilibrium anymore. We refer to this minimum cost as the margin of robustness of the transportation network and provide explicit formulas for it. Our results extend those in the M.Sc. thesis [5].

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This research was carried on within the framework of the MIUR-funded *Progetto di Eccellenza* of the *Dipartimento di Scienze Matematiche G.L. Lagrange*, CUP: E11G18000350001, and was partly supported by the *Compagnia di San Paolo* and the Swedish Research Council.

Stability of the CTM and of its continuous-time generalizations has been studied in [6] and [7], [8], [9], [10], respectively. Equilibrium selection and optimal control problems for these models have been studied in [11], [12]. The results in this paper should be compared with those in [13], [14], [15], [16], where the margin of resilience of dynamical flow networks controlled by distributed feedback policies is analyzed, also in the presence of cascading failure mechanisms. Differently from those works, this paper focuses on robustness of the dynamical flow network with a prescribed routing matrix. This makes the analysis in this paper different from other previous approaches in, e.g, [17], [18] where the robustness of dynamically assigned routing is studied.

The outline of the paper is as follows: The rest of this section is devoted to some basic notation that will be used throughout the paper. In Section II we present the dynamical model for transportation networks, and specify what we mean with a free flow equilibrium. In Section III we study how perturbations in the exogenous inflows and maximum flow capacities affect the free flow equilibrium, and exploit the similarity between those two perturbations. We also provide an example of the robustness of a system-optimal equilibrium, and show that it may not be the most robust equilibria. In Section IV, perturbations in the routing are considered.

A. Notation

We let $\mathbb{R}_{(+)}$ denote the set of (non-negative) reals. For finite set $\mathcal{A}$, we denote $\mathbb{R}^{\mathcal{A}}_{(+)}$ the set of (non-negative) vectors indexed by the elements of the set $\mathcal{A}$. Inequalities between vectors means element-wise inequalities, i.e., two vectors $a, b \in \mathbb{R}^{\mathcal{A}}$, $a \leq b$ respectively $a < b$, means that $a_i \leq b_i$ respectively $a_i < b_i$ for all $i \in \mathcal{A}$. We use $\mathbf{1}$ to denote the all 1 vector with appropriate dimension, and $\mathbf{1}^{(i)}$ denotes a vector with all zeros, expect for a 1-element at position i. The indicator function $\delta_i^{(j)}$ is 0 when $i \neq j$, and 1 when $i = j$.

II. Model

We model the topology of the transportation network as a directed multi-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the multi-set of directed links. We will assume that $\mathcal{G}$ does not contain any self-loops, while allowing for the presence of parallel links (hence the prefix "multi"). The links $i \in \mathcal{E}$ will represent *cells*. For a cell $i \in \mathcal{E}$, we will let $\mathcal{E}_i^+ \subset \mathcal{E}$ denote the set of out-neighboring cells, i.e., outflow from one cell i, will either leave the network or proceed to one of the cells in $\mathcal{E}_i^+$.

To each cell $i \in \mathcal{E}$, two functions are assigned, the *demand* and *supply* functions. They both depend on the traffic volume in the cell. The demand function, denoted by $\varphi_i(x_i)$, is non-decreasing, concave, and such that $\varphi_i(0) = 0$. The supply function, denoted $\sigma_i(x_i)$, is assumed to be Lipschitz-continuous, non-increasing, concave and such that $\sigma_i(x_i) = 0$ for all $x_i \geq x_i^{\max}$, where $x_i^{\max} > 0$. From the demand and supply function, the cell's maximum flow capacity C_i can determined as

$$C_i = \max_{x_i \geq 0} \left(\min(\varphi_i(x_i), \sigma_i(x_i))\right) . \quad (1)$$

The vector of all cells' capacities is denoted by $C \in \mathbb{R}_+^{\mathcal{E}}$.

To a subset of cells $\mathcal{I} \subseteq \mathcal{E}$ in the transportation network, there may be an exogenous inflow of traffic flow. We denote the vector of such exogenous inflows by $u \in \mathbb{R}_+^{\mathcal{E}}$, with the property that $u_i = 0$ for every $i \in \mathcal{E} \setminus \mathcal{I}$.

To model how the traffic propagate through network, we introduce the routing matrix $R \in \mathbb{R}_+^{\mathcal{E} \times \mathcal{E}}$. An entry R_{ij} of the routing matrix represents the fraction of the outflow from cell i that will proceed directly to cell j. The routing matrix R has to adhere to the topology, i.e., if $R_{ij} > 0$ then in the multigraph $\mathcal{G}$ the head node of link i and the tail node of link j must coincide. Note that, since $\mathcal{G}$ contains no self-loops, this implies that $R_{ii} = 0$ for all $i \in \mathcal{E}$. Moreover, the routing matrix is assumed to be sub-stochastic, i.e., $R\mathbf{1} \leq \mathbf{1}$. This in order to obey the conservation of mass, so that no flow is created from the outflow from one cell. If $\sum_j R_{ij} < 1$ for a cell i, it means that a fraction of the outflow, given by $1 - \sum_j R_{ij}$, will leave the network directly when flowing out of i. We will further refer to such cells as *sink* cells and denote their set as

$$\mathcal{S} = \left\{ i \in \mathcal{E} \mid \sum\nolimits_j R_{ij} < 1 \right\} .$$

A routing matrix is said to be *out-connected* if for every cell $i \in \mathcal{E} \setminus \mathcal{S}$ there exist a sink cell $j \in \mathcal{S}$ and a positive integer l such that $(R^l)_{ij} > 0$. We denote the set of out-connected routing matrices by $\mathcal{R}$. For a subset of cells $\mathcal{I} \subseteq \mathcal{E}$, we also say that the routing matrix is in-connected from $\mathcal{I}$, if for every cell $j \in \mathcal{E} \setminus \mathcal{I}$ there exist some cell $i \in \mathcal{I}$ and a positive integer l such that $(R^l)_{ij} > 0$.

We are now ready to describe the dynamical flow network model. We shall denote the traffic volume in and the outflow from a cell $i \in \mathcal{E}$ by x_i and z_i, respectively. Let $\mathcal{X} = \mathbb{R}_+^{\mathcal{E}}$ and let $x = (x_i)$ and $z = (z_i)$ in $\mathcal{X}$ denote the vectors of all traffic volumes and of all outflows, respectively. Let also $\varphi(x) = (\varphi_i(x_i))$ and $\sigma(x) = (\sigma_i(x_i))$ be the vectors of demand and supply functions. The dynamics of the transportation network can then be described by the ODE

$$\dot{x} = u - (I - R^T)z , \quad (2)$$

where

$$z = f(x) , \quad (3)$$

for a Lipschitz continuous function $f : \mathcal{X} \to \mathbb{R}_+^{\mathcal{E}}$ satisfying the demand constraint

$$f(x) \leq \varphi(x) , \quad (4)$$

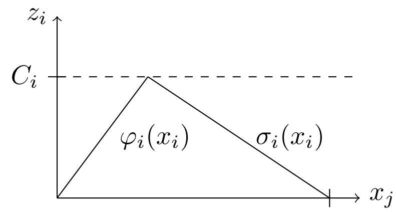

Fig. 1. An example of a fundamental diagram with affine supply and demand functions, as in Daganzo's cell transmission model.

and the supply constraint

$$u + R^T f(x) \leq \sigma(x) , \quad (5)$$

for every $x \in \mathcal{X}$. We will also assume that the outflow is equal to the demand when the supply constraints in (5) are not active. Formally, let us define the *freeflow region* $\mathcal{F} \subseteq \mathcal{X}$ as

$$\mathcal{F} = \{x \in \mathcal{X} \mid u + R^T \varphi(x) \leq \sigma(x)\} .$$

Then, we shall assume that

$$f(x) = \varphi(x) , \qquad x \in \mathcal{F} . \quad (6)$$

In summary, the results in this paper apply to models of dynamical flow networks in the form (2)–(3) satisfying (4)–(6). As explained in [10], this allows for several possible dependancies of the outflow vector z on x outside the free-flow region, i.e., when some supply constraints in (5) are active. For example, a common way to determine the outflow is to consider a first-in-first-out (FIFO) rule as in Example 1 below. For other models, see, e.g., [7] and [10].

Example 1 (FIFO rule): Let the outflow from a cell i be given by

$$f_i(x) = \kappa_i(x)\varphi_i(x_i) , \quad (7)$$

where

$$\kappa_i(x) = \sup \left\{ \xi \in [0, 1] \mid \max_{k \in \mathcal{E}_i^+} \left(\xi \sum_{h \in \mathcal{E}} R_{hk}\varphi_h(x_h) - \sigma_k(x_k) \right) \leq 0 \right\} \quad (8)$$

This means that if one of the immediately downstream cells are congested, it will affect the outflow from the upstream cell to all its immediately downstream cells.

Observe that, for all $x \in \mathcal{X}$ and $i \in \mathcal{E}$, we have that $\gamma_i(x) \leq 1$ so that (7) guarantees that the demand constraint (4) is satisfied. Moreover, (8) guarantees that the supply constraint (5) is also fulfilled. Finally, observe that in the freeflow region, i.e., for $x \in \mathcal{F}$, we have $\kappa_i(x) = 1$ in (7) for all cells $i \in \mathcal{E}$, so that (6) is satisfied as well.

Remark 1 (Connection to Daganzo's CTM): Let $L_i > 0$ be the length of a cell i and let the demand and supply functions be

$$\varphi_i(x_i) = \frac{v_i}{L_i} x_i , \qquad \sigma_i(x_i) = \frac{-w_i x_i + x_i^{\text{jam}} w_i}{L_i} ,$$

as in Fig. 1. Here, the constants $v_i > 0$ and $w_i > 0$ are referred to as the free-flow speed and the shock-wave speed, respectively. This diagram is referred to as the fundamental diagram for traffic flows. If we discretize the dynamics with an Euler-discretion with a step-size $0 < h \leq \frac{L_i}{x_i}$ for all $i \in \mathcal{E}$, and these supply and demand functions, we obtain the cell transmission model as was proposed by Daganzo in [1], [2].

Throughout the paper, we will assume that the routing matrix is out-connected. Under this assumption the routing matrix has spectral radius strictly less than 1, see e.g., [19, Theorem 2]. This implies that the matrix $I - R^T$ is invertible, and we will denote its inverse by

$$H = (I - R^T)^{-1} . \tag{9}$$

Several topological properties of the routing matrix R affect the matrix H, a few of which are gathered in the following result proven in Appendix A. Let an R-path be a string of cells $(i_1, i_2, \dots, i_l) \in \mathcal{E}^l$ such that $i_h \neq i_k$ for all $h \neq k$ and $R_{i_h, i_{h+1}} > 0$ for every $h = 1, 2, \dots, l-1$, and let an R-cycle be an R-path such that the head node of i_l coincides with the tail node of i_1.

Lemma 1: The matrix H has the following properties:

i) For all i, $H_{ii} \geq 1$ with equality if and only if i lies in no R-cycle.
ii) For all $i \neq j$, $H_{ij} \leq H_{ii}$ with equality if and only if all R-paths from j to $\mathcal{S}$ pass through i.
iii) $H_{ij} > 0$ if there exists an R-path from cell j to cell i.
iv) If R is acyclic, then there exists a permutation p such that $H_{i,p(j)} = 0$ for all $j > i$.

If the dynamical system in (2) and (3) together with the constraints (7)–(8) has an equilibrium, the equilibrium flows z^* are given by

$$z^* = (I - R^T)^{-1} u = Hu .$$

Moreover, we introduce the set of inflow, routing matrices and capacities, such that a freeflow equilibrium exists, the *feasibility region* defined as

$$\Lambda = \{(u, R, C) \in \mathbb{R}_+^{\mathcal{E}} \times \mathcal{R} \times \mathbb{R}_+^{\mathcal{E}} \mid Hu = (I - R^T)^{-1} u \leq C\} . \tag{10}$$

Let $x^* \in \mathcal{X}$ denote the equilibrium to the dynamics. Observe that $(u, R, C) \in \Lambda$ does not necessary imply that $x^* \in \mathcal{F}$, it will depend on the initial condition $x(0)$. However, for given $(u, R, C) \in \Lambda$, there exists a equilibrium point $x^* \in \mathcal{F}$, and this equilibrium point is unique and stable, as the following proposition states.

Proposition 1: For a given $(u, R, C) \in \Lambda$ with strictly increasing demand functions $\varphi_i(x_i)$, there exists a unique equilibrium point in $x^* \in \mathcal{F}$ to the dynamics given by (2), (3), (7), and (8). Moreover, this equilibrium point is locally asymptotically stable.

Proof: Since the demand functions are strictly increasing, the traffic volumes at the flow equilibrium z^* when the the equilibrium is in free-flow is given by

$$x_i^* = \varphi_i^{-1}(z_i^*) .$$

The local asymptotical stability follows from [10]. ∎

While other equilibria to the dynamical system (2)–(3) together with the constraints (7)–(8) exist, the equilibrium point $x^* \in \mathcal{F}$ is the desired one, since it achieves the same throughput in the network with less traffic present in the network compared to any other equilibria. Moreover, as shown in [20], it is most efficient to operate the transportation network at this equilibrium point, rather than trying to steer some time-varying trajectories. Therefore this equilibrium will be the focus of the studies in this paper.

III. Inflow and Capacity Perturbations

In this section, we study how robust different routing matrices are with respect to perturbations on the inflows and on the link capacities. Specifically, we are searching for perturbed exogenous inflow vectors $\tilde{u}$ or perturbed flow capacity vectors $\tilde{C}$ such that the transportation network does not have a free-flow equilibrium anymore.

The capacity perturbations are occurring in the supply or demand functions, such that the capacities given by (1) decreases. Let the vector $\tilde{C} \in \mathbb{R}_+^{\mathcal{E}}$ be the perturbed capacity vector, and $\hat{C} \in \mathbb{R}_+^{\mathcal{E}}$ the magnitude of the capacity drops. Then,

$$\tilde{C} = C - \hat{C} ,$$

where it is assumed that C is such that $(u, R, C) \in \Lambda$. In the same manner as the capacity perturbations, we denote the perturbed inflow vector $\tilde{u} \in \mathbb{R}_+^{\mathcal{E}}$ and let $\hat{u} \in \mathbb{R}_+^{\mathcal{E}}$ be the perturbation. Hence the perturbed inflow vector is given by

$$\tilde{u} = u + \hat{u} ,$$

where it is assumed that $(u, R, C) \in \Lambda$.

For a given capacity and flow perturbation, we introduce the cost of perturbation, $\gamma(\hat{u}, \hat{C})$, as

$$\gamma(\hat{u}, \hat{C}) = \sum_{i \in \mathcal{E}} \Psi_i(\hat{C}_i) + \sum_{i \in \mathcal{S}} \Phi_i(\hat{u}_i) ,$$

where $\Psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and convex cost functions for perturbing the flow capacities of cells $i \in \mathcal{E}$ and $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and convex cost functions for perturbing the exogenous inflows to cells $i \in \mathcal{I}$. We moreover assume that $\Psi_i(0) = 0$ for all $i \in \mathcal{E}$, and $\Phi_i(0) = 0$ for all $i \in \mathcal{I}$.

We say that the transportation network is robust to perturbations up to a given cost k, if for all $\hat{C}$, $\hat{u}$ such that $\gamma(\hat{C}, \hat{u}) \leq k$, it holds that $(\tilde{u}, R, \tilde{C}) \in \Lambda$. We can then define the network's robustness margin as follows

Definition 1: The transportation network's robustness margin is given by

$$\Gamma = \inf\{\gamma(\hat{C}, \hat{u}) \mid (\tilde{u}, R, \tilde{C}) \notin \Lambda\} .$$

The definition above means that: (i) under any perturbations with a cost less than the robustness margin Γ the perturbed dynamics will still have a free-flow equilibrium; (ii) there exist perturbations with a cost higher than, but arbitrarily close to, the margin of robustness Γ such that the perturbed dynamics do not admit a free-flow equilibrium.

Proposition 2: A transportation network's robustness margin is given by

$$\Gamma = \min_{i\in\mathcal{E}} \bar{\gamma}_i \,, \tag{11}$$

where

$$\begin{aligned} \bar{\gamma}_i = & \min_{\substack{0\le \hat{C}_i \le C_i \\ \hat{u}\in\mathbb{R}_+^{\mathcal{I}}}} \Psi_i(\hat{C}_i) + \sum_{j\in\mathcal{I}} \Phi_j(\hat{u}_j) \\ & \text{subject to} \quad z_i^* + \sum_{j\in\mathcal{I}} H_{ij}\hat{u}_j - C_i + \hat{C}_i = 0 \,. \end{aligned} \tag{12}$$

Proof: To compute the network's robustness margin, we have to search for the perturbation with the smallest cost, such that inequality constraint in (10) is violated. This is equivalent to solving the following optimization problem

$$\begin{aligned} \min_{\hat{C},\hat{u}} & \sum_{i\in\mathcal{E}} \Psi_i(\hat{C}_i) + \sum_{i\in\mathcal{I}} \Phi_i(\hat{u}_i) \\ \text{subject to} \quad & \max_i (H\tilde{u} - \tilde{C})_i \ge 0 \,, \\ & \hat{u} \ge 0 \,, \quad 0 \le \hat{C} \le C \,, \\ & \hat{u}_i = 0 \,, \quad \forall i \in \mathcal{E}\setminus\mathcal{I} \,. \end{aligned}$$

While the optimization problem above is not necessarily convex, monotonicity of the cost functions Ψ_i and Φ_i implies that an optimal solution satisfies

$$\sum\nolimits_j H_{ij}\tilde{u}_j - \tilde{C}_i = 0 \tag{13}$$

for at least one cell i. Moreover, if (13) is satisfied by an optimal feasible solution, then necessarily $\hat{C}_j = 0$ for all $j \neq i$ because of monotonicity of the cost function Ψ_j. The whole network's robustness margin is then given by (11). ■

A. Linear Cost Functions

In the special case of linear cost functions, i.e., $\Psi_i(\hat{C}_i) = \alpha_i\hat{C}_i$ and $\Phi_i(\hat{u}_i) = \beta_i\hat{u}_i$, where $\alpha_i > 0$ and $\beta_i > 0$, the problem of computing a cell's i robustness margin in (12), given that $z_i^* > 0$, can be rewritten as

$$\min_{\hat{u}} \alpha_i(C_i - z_i^*) + \sum_{j\in\mathcal{I}} (\beta_j - \alpha_i H_{ij})\hat{u}_j \,. \tag{14}$$

If $\beta_j > \alpha_i H_{ij}$ for all $j \in \mathcal{I}$, then $\hat{u} = 0$. This means that the perturbation with the smallest cost such that a free flow equilibrium does not exist anymore will be a capacity perturbation. The cell's margin of robustness is then $\bar{\gamma}_i = C_i - z_i^*$.

On the other hand, if $\beta_j < \alpha_i H_{ij}$ for some $j \in \mathcal{I}$, then the perturbation is instead an inflow perturbation such that

$$\hat{u}_j = \frac{C_i - z_i^*}{H_{ij}} \,,$$

where

$$j \in \operatorname*{argmin}_{k\in\mathcal{I}} \beta_k - \alpha_k H_{kj} \,.$$

For the special case when $\beta_j = \alpha_i H_{ij}$ for all $j \in \mathcal{S}$, either a capacity reduction such that $\hat{C}_i = C_i - z_i^*$ or a inflow perturbation such that

$$\hat{u}_j = \frac{C_i - z_i^*}{H_{ij}} \,,$$

for one or several $j \in \mathcal{I}$ such that $H_{ij} > 0$ will cause the perturbed equilibrium to be outside the free-flow region.

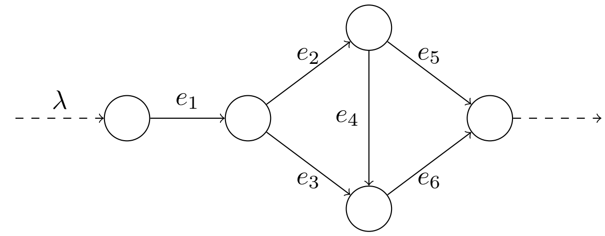

Fig. 2. The network in Section III-B. Cell e_1 is a source cell, so $\mathcal{I} = \{e_1\}$. Cells e_5 and e_6 are sink cells, so $\mathcal{S} = \{e_5, e_6\}$.

Remark 2: If the transportation network is a line graph where the first cell has a strictly positive inflow, it follows from Lemma 1 that, $H_{ij} = 1$ for all $j \le i$, and $H_{ij} = 0$ for all $j \ge i$. Suppose that $\alpha_i = \xi$, for some $\xi > 0$, and $\beta_i = \alpha_i$ for all $i \in \mathcal{E}$. Then, given that the exogenous inflow to first is strictly positive, the optimization problem in (14), reads

$$\min_{\hat{u}} \xi(C_i - z_i^*) + \sum_{j>i} \hat{u}_j \,.$$

In this case it will not matter for a cell's robustness if the perturbation is done by a capacity reduction or an upstream inflow perturbation.

Remark 3: Suppose that the transportation network is acyclic, and $\alpha_i \le \beta_j$ for all $j \in \mathcal{I}$ and all $i \in \mathcal{E}$, and $z_i^* > 0$ for all $i \in \mathcal{E}$. Then the coefficients of all $\hat{u}$ in (14) will be non-negative, due to the fact that $H_{ij} \ge 0$ as stated in Lemma 1. This means that an optimal solution is $\hat{u} = 0$ and a capacity perturbation can always be used to determine the network's robustness margin.

B. Numerical Example

In this section we give an example showing that a system optimal traffic assignment may not be the most robust one. To illustrate this, we consider a network with six cells depicted in Fig. 2, where cell e_1 accepts exogenous inflows, and cell e_5 and e_6 are sink cells. For each of the cells, we let the demand function be $\varphi_i(x_i) = C_i(1 - e^{-x_i})$, where $C_i > 0$ is the cell's capacity.

For those demand functions, as shown in [14], the corresponding delay functions for given flows z_i are, under the assumption that all cells are of unit lengths,

$$\tau_i(z_i) = \frac{1}{z_i}\log\left(\frac{C_i}{C_i - z_i}\right) \,.$$

Moreover, let B be the node-link incidence matrix to network in Fig. 2. The system-optimal traffic assignment is then computed by

$$\begin{aligned} \underset{z}{\text{minimize}} \quad & \sum_i z_i\tau_i(z_i) \\ \text{subject to} \quad & Bf = \lambda(\mathbf{1}^{(1)} - \mathbf{1}^{(5)}) \,, \\ & f \ge 0 \,, \end{aligned} \tag{15}$$

where λ is the exogenous inflow to cell e_1.

We let the cells' capacities be $C_{e_1} = 10$, $C_{e_2} = C_{e_4} = 4$ and $C_{e_3} = C_{e_5} = C_{e_6} = 2$. If we let $\lambda = 2$, the solution to (15) is $z^* = (2, 1.49, 0.51, 0.35, 1.13, 0.87)$. This equilibrium yields a routing matrix such that $R_{e_1,e_2} = 0.74, R_{e_1,e_4} = 0.26$, and $R_{e_2,e_4} = 0.26, R_{e_2,e_5} = 0.74$.

We let the cost functions be linear, with $\alpha_i = 1$ for all cells i, and $\beta_{e_1} = 1$. By utilizing the fact from Remark 3, the cells' robustness margins are given by theirs residual capacities, so $\bar{\gamma} = (8, 2.51, 1.49, 3.65, 0.87, 1.13)$, and the whole network's robustness margin is 0.87.

If, we instead let $\alpha_i = 2$ but $\beta_{e_1} = 1$, i.e., in inflow perturbation is less costly than a capacity reduction, then the network's margin of robustness is given by $\bar{\gamma} = (8, 3.38, 2.98, 7.29, 1.53, 2.27)$. In this case, the smallest perturbation causing the network to not have a free flow equilibrium anymore is an increase of the inflow λ with 1.53.

Let us now compare the robustness margin values above with a non-system optimal assignment, by letting $R_{e_1,e_2} = R_{e_1,e_3} = 0.5$ and $R_{e_2,e_4} = 1$, i.e, half of the flow take the upper path, half the lower path. For the first set of cost functions, i.e., $\alpha_i = 1$ for all cells i, and $\beta_{e_1} = 1$, the cell's margin of robustness is $\bar{\gamma} = (8, 2, 1, +\infty, 1, 1)$. Hence the the network's margin of robustness is 1, which is slightly higher compared to the system optimal assignment, 0.87.

For the second set of cost functions, the cell's robustness margin is $\bar{\gamma} = (8, 6, 2, +\infty, 2, 2)$. In this case, the perturbation is either an inflow increase of two units or a capacity reduction at cell e_3, e_5, or e_6 by one unit, which will have the cost of 2. Hence, the network's robustness margin is 2.

From the results above, we can see that there exists traffic assignments that are more robust compared to the system optimal one. How one should do the tradeoff between optimality and robustness is a topic for further research.

IV. Routing Perturbations

In this section we consider perturbations that will affect the turning ratios of a single cell. We denote this cell $i \in \mathcal{E}$, and let $r_j^{(i)} \in \mathbb{R}$ denote the perturbation to a routing ratio, i.e., the addition to the fraction of the flow that will go from cell i to cell j.

Moreover, we assume that the vector $r^{(i)}$ is only supported on the out-neighboring cells of cell i. The perturbed routing matrix $\tilde{R}^{(i)}$ is then given by

$$\tilde{R}^{(i)} = R + \mathbf{1}^{(i)}(r^{(i)})^T \,. \tag{16}$$

The topic of interest in this section is to investigate when the perturbed routing matrix yields a feasible perturbation, i.e., when

$$(u, \tilde{R}^{(i)}, C) \in \Lambda \,.$$

Proposition 3: Let z^* be a flow equilibrium for a given $(u, R, C) \in \Lambda$. Assume that a routing matrix R is perturbed according to (16). Then $(u, \tilde{R}^{(i)}, C) \in \Lambda$ if and only if for all $k \in \mathcal{E}$ it holds that

$$\frac{\sum_j H_{kj}\tilde{R}_{ij}^{(i)} - H_{ki} + \delta_k^{(i)}}{C_k - z_k^*} \leq \frac{H_{ii} - \sum_j H_{ij}\tilde{R}_{ij}^{(i)}}{z_i^*} \,. \tag{17}$$

Proof: The Sherman-Morrison formula [21] states that for an invertible matrix A and two column vectors v, w, the matrix $A + vw^T$ is invertible if and only if $1 + w^T A^{-1} v \neq 0$ and its inverse is given by

$$(A + vw^T)^{-1} = A^{-1} - \frac{A^{-1}vw^TA^{-1}}{1 + w^TA^{-1}v} \,.$$

Hence,

$$\tilde{H}^{(i)} = (I - (\tilde{R}^{(i)})^T)^{-1} = (I - (R^T + r^{(i)}\left(\mathbf{1}^{(i)}\right)^T))^{-1} =$$
$$(I - R^T - r^{(i)}\left(\mathbf{1}^{(i)}\right)^T)^{-1} = H + \frac{Hr^{(i)}\left(\mathbf{1}^{(i)}\right)^T H}{1 - \left(\mathbf{1}^{(i)}\right)^T Hr^{(i)}} \,.$$

Since, we are interested in the new equilibrium flows

$$\tilde{H}^{(i)}u = z^* + \frac{Hr^{(i)}z_i^*}{1 - \left(\mathbf{1}^{(i)}\right)^T Hr^{(i)}} \,. \tag{18}$$

We now observe that

$$r^{(i)} = (\tilde{R}^{(i)})^T\mathbf{1}^{(i)} - R^T\mathbf{1}^{(i)} \,, \tag{19}$$

and hence

$$\begin{aligned} 1 - \left(\mathbf{1}^{(i)}\right)^T Hr^{(i)} &= 1 - (\mathbf{1}^{(i)})^T H((\tilde{R}^{(i)})^T - R^T)\mathbf{1}^{(i)} = \\ &1 - (\mathbf{1}^{(i)})^T H((\tilde{R}^{(i)})^T - (I - H^{-1}))\mathbf{1}^{(i)} = \\ &1 + H_{ii} - \sum_j H_{ij}\tilde{R}_{ij}^{(i)} - 1 = H_{ii} - \sum_j H_{ij}\tilde{R}_{ij}^{(i)} \,. \end{aligned} \tag{20}$$

For the equilibrium to be in free-flow, it must hold that

$$\tilde{H}^{(i)}u \leq C \,.$$

Combining this inequality with (18), (19), and (20) yields

$$\frac{H(\tilde{R}^{(i)})^T\mathbf{1}^{(i)} - HR^T\mathbf{1}^{(i)}}{H_{ii} - \sum_j H_{ij}\tilde{R}_{ij}^{(i)}} z_i^* \leq C - z^* \,.$$

For a specific row k, it must then hold that

$$\frac{\sum_j H_{kj}\tilde{R}_{ij}^{(i)} - H_{ki} + \delta_k^{(i)}}{C_k - z_k^*} \leq \frac{H_{ii} - \sum_j H_{ij}\tilde{R}_{ij}^{(i)}}{z_i^*} \,,$$

which concludes the proof. ■

We make a few observations about eq. (17). First, observe that the right hand side will always be non-negative since Lemma 1 together with the fact that $\sum_j \tilde{R}_{ij} \leq 1$ gives that

$$H_{ii} - \sum\nolimits_j H_{ij}\tilde{R}_{ij}^{(i)} \geq H_{ii} - \max_{j \neq i} H_{ij} \geq 0 \,.$$

Note however that the right hand side numerator needs to be positive for there to be a $\tilde{H}^{(i)}$, as per the conditions stated in the Sherman-Morrison formula. Moreover, the right hand side is upper bounded by H_{ii}/z_i^*. In the case where z_i^* is equal to zero, the perturbed cell can not be reached by any cell in $\mathcal{I}$ and as such routing perturbations on that cell have no effect on the equilibrium. In the case(s) where $C_k - z_k^*$ is equal to zero, it is necessary for the left hand side numerator to be non-positive for the condition to hold.

We say that a cell i is robust to routing perturbations, if any $r^{(i)}$ supported on the out-neighbouring cells of i, will still be such that $(u, \tilde{R}^{(i)}, C) \in \Lambda$. From the results above, we can state a proposition when a given cell i is robust to all routing perturbations.

Corollary 1: A given cell i is robust to routing perturbations if, for all $k \in \mathcal{E}$,

$$\max_{j \in \mathcal{E}_i^+} \left\{ \frac{H_{kj}}{C_k - z_k^*} + \frac{H_{ij}}{z_i^*} \right\} - \frac{H_{ki} - \delta_k^{(j)}}{C_k^* - z_k^*} - \frac{H_{ii}}{z_i^*} \leq 0 \, ,$$

where $\mathcal{E}_i^+$ are the out-connected cells from cell i.

V. Conclusions

We have presented a framework for robustness analysis in transportation networks. We do this by defining the network's robustness margin to inflow and capacity perturbations and find the cost of smallest perturbation such that the network does not have a free-flow equilibrium anymore. We also investigate how a perturbation in the predefined routing from a cell will affect the existence of a free-flow equilibrium. A direction of future research is to investigate more deeply how one can do traffic assignments that are both close to optimal and robust, and how one should tradeoff between those two objectives. As shown in Section III-B it is not always the case that the optimal assignment is also the most robust one.

Acknowledgements

The authors wish to thank Christian Rosdahl from Lund University for insightful discussions and active collaboration.

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Appendix

A. Proof of Lemma 1

To prove the claims, we will make use of the fact that matrix H in (9) can be written as a power series,

$$H = \left(I - R^T\right)^{-1} = I + R^T + (R^T)^2 + \dots \qquad (21)$$

i) From the formula above it is clear that $H_{ii} \geq 0$. In order to show the the equality part, recall that $\left(R^l\right)_{ii}$ is the aggregate weight of all length-l walks starting and ending in cell i. Hence it will be zero for all $l > 0$, if i lies in no R-cycle.

ii) To prove this part, we observe that R can be interpreted as a random walk, where the states in $i \in \mathcal{I}$ are such that an absorbing state is reached with probably $1 - \sum_j R_{ij}$. Then H_{ii}, is the number of expected visits in i before hitting the absorbing state, given that the walk starts from i. In the same way, H_{ij} is the number of expected visits in i before hitting the absorbing state, given that the walk start from j. From this interpretation, it can be seen that $H_{ij} \leq H_{ii}$, since every walk starting from j and hitting i, will also be counted as a walk starting from i. On the other hand, all walks starting from j, will only hit i, if there is no other paths to the absorbing state, which proves the second part of the statement.

iii) Follows from the fact that $(R^l)_{ji}$ is the aggregate weight of all length-l walks from cell j to cell i.

iv) When the graph is acyclic, there exists an topological ordering among the nodes see, e.g., [22, Theorem 22.12]. Hence it is possible to permute the node ordering such that R is upper triangular, and hence R^T lower triangular. Together with (21) it can be seen that if R is upper triangular H will be lower triangular.

■