On Generalized Proportional Allocation Policies for Traffic Signal Control ⋆

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Abstract: The fast-increasing demand and relatively slow growth of infrastructure capacity are providing a strong motivation for research in real-time urban traffic controls that make the best use of novel sensing in order to increase efficiency and resilience of the transportation system. In our contribution, we focus on a class of dynamic feedback traffic signal control policies that are based on a generalized proportional allocation rule. The proposed traffic signal controls are decentralized (they make use of local information only), scalable (they are independent of the network size and topology), and universal (they do not rely on any information about external inflows or turning ratios). In spite of their fully distributed nature, we prove that such control policies achieve a global objective, maximum throughput, in that they stabilize the urban traffic network whenever possible under the given capacity constraints.
The traffic model we consider consists in a network of interconnected vertical queues with deterministic dynamics driven by physical laws (conservation of mass and preservation of non-negativity of the traffic volumes) as well as scheduling constraints (described as a set of phases, each phase consisting in a subset of lanes that can be be given green light simultaneously). This results in a differential inclusion for which we prove existence and, in the special case of orthogonal phases, uniqueness of continuous solutions via a generalization of the reflection principle. Stability is then proved by interpreting the generalized proportional allocation controllers as minimizers of a certain entropy-like function that is then used as a Lyapunov function for the closed-loop system.

Keywords: Distributed Traffic Signal Control, Nonlinear Control, Dynamical Flow Networks

1. INTRODUCTION

In today's transportation systems, traffic signal control plays a key role for maximizing throughput and reducing congestion. In order to design traffic signal controllers, one approach is to used fix-timed controllers, as proposed in e.g., Miller (1963). To achieve more robustness under changing arrival rates, constantly re-tuned controllers have been developed for several cities, for example SCOOT, see Bretherton et al. (1998). With the recent development of cheap and reliable sensors, the stage is now set for the introduction of feedback-based traffic light controllers.

In queuing networks, research on stabilizing feedback controllers has been ongoing for some decades. While the original back-pressure controller presented in Tassiulas and Ephremides (1992) is not directly applicable to road traffic networks, [1] recent works such as Varaiya (2013b), Varaiya (2013a) and Wongpiromsarn et al. (2012) have adapted it to the purpose by giving the back-pressure controller exogenous information about the turning ratios. However, the turning ratios are often difficult predict with high accuracy. In Gregoire et al. (2014) the dependency of the turning ratios is avoided by letting the back-pressure controller check if the incoming queue-lengths are above a certain threshold level and react to that. However, this modification leads to an unspecified shrinkage of the network's stability region. In Le et al. (2015) a solution is proposed on how to construct a back-pressure controller relaying of estimates of the turning ratios.

In this paper, we study feedback traffic signal control policies that are based on a generalized proportional allocation rule. These controls do not require any information about the turning ratios or the external arrival rates (a property referred to as universality), they are independent of the network size and topology (scalability), and make use of local information only (decentralized). [2] The stability analysis of the proportional allocation policy for data networks was first done in Massoulié (2007) while Walton (2014) studies stability in a multi-commodity setting.

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[1] The controller assumes that vehicles are distinguishable by their destination, and cannot for instance handle when a lane is used both for right turns and vehicles that want to proceed straight forward.

[2] In fact, as compared to the back-pressure controllers, the generalized proportional allocation controllers proposed here requires state information about the incoming lanes, while the back-pressure controller requires information about the outgoing lanes as well.

We focus on the continuous-time traffic network dynamical model first studied in Savla et al. (2013), Savla et al. (2014), and Nilsson et al. (2015), and extend the results proved there in several directions. First, while the analysis in Savla et al. (2013) and Savla et al. (2014) was restricted to acyclic network topologies and built on monotone flow networks techniques (c.f. Como et al. (2013, 2015); Lovisari et al. (2014); Como (2017)), we consider here general network topologies for which the resulting closed-loop traffic network dynamics are not monotone. This requires the use of different techniques to establish stability, in particular suitable entropy-like Lyapunov functions, similar to those used in Massoulié (2007) for data networks and adapted to traffic networks in Nilsson et al. (2015).

Second, in contrast to Nilsson et al. (2015) where stability of generalized proportional allocation policies was studied in a setting where only one incoming lane to each junction can receive green light simultaneously, we handle the general case where several lanes can receive green light simultaneously in each phase. Far from being trivial, this generalization implies several additional challenges, in particular related to the fact that the resulting traffic network dynamics can no longer be expressed as a regular (Lipschitz-continuous) differential equation, for which existence and uniqueness of solutions are standard facts. This problem results from the fact that, if there are phases that contain more than one lane, then the generalized proportional allocation controller can assign green light to empty lanes, so that the dynamics when some lanes are empty needs to be properly modified in order to guarantee that traffic volumes remain nonnegative over time (equivalently, the nonnegative orthant is an invariant set).

In this paper, we handle this issue by first formulating the closed-loop controlled traffic network dynamics as a differential inclusion that incorporates all the mass conservation, non-negativity, and traffic signal control constraints. This is quite a natural model choice for traffic queues and has previously been proved to be the fluid limit of queueing networks, see e.g. Massoulié (2007), as well as traffic networks, see Muralidharan et al. (2015). While existence of continuous solutions then follows from general results on differential inclusions, one of our main contributions consists in proving existence and uniqueness of solutions for the case where the phases are locally orthogonal (equivalently, that each lane belongs to at most one local phase): this result is stated in Theorem 1.

Another benefit of the chosen differential inclusion approach is that the stability result holds for every absolutely continuous solution of the differential inclusion. Such stability analysis includes additional challenges with respect to the case addressed in Nilsson et al. (2015): in particular, we use an argument based on LaSalle's principle.

We end this section by introducing some basic notation. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+$ the set of nonnegative reals. For finite sets $\mathcal{A}$ and $\mathcal{B}$, let $|\mathcal{A}|$ denote the cardinality of $\mathcal{A}$ and $\mathbb{R}^{\mathcal{A}}$ the space of real-valued vectors whose elements are indexed by $\mathcal{A}$. Let $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ denote a directed multigraph where $\mathcal{E}$ is the set of directed links and $\mathcal{V}$ is the set of vertices or nodes. For each link $e = (i, j) \in \mathcal{E}$, let $\tau_e = j \in \mathcal{V}$ denote the head of the link e and $\sigma_e = i \in \mathcal{V}$ the tail of the link e.

2. TRAFFIC NETWORK DYNAMICS MODEL

We model the traffic network as a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, m\}$ is the set of nodes that represent signalized junctions and $\mathcal{E} = \{1, \dots, n\}$ is the set of links that represent lanes. To each lane, two nonnegative variables are associated: the *traffic volume* $x_i(t)$ and the *outflow* $z_i(t)$. While we assume no a priori upper bound on the traffic volume $x_i \geq 0$, we will assume that the outflow is upper bounded by a constant *flow capacity*, $c_i > 0$, so that $0 \leq z_i \leq c_i$ for all $i \in \mathcal{E}$. Traffic volumes, outflows and capacities for each lane are all stacked up into vectors $x(t) \in \mathbb{R}_+^{\mathcal{E}}$, $z(t) \in \mathbb{R}_+^{\mathcal{E}}$ and $c \in \mathbb{R}_+^{\mathcal{E}}$, respectively. Moreover the notation $C = \operatorname{diag}(c)$ is used for the diagonal matrix with the diagonal c. The non-negativity constraints on the traffic volume can then be compactly written as

$$x \geq 0\,, \tag{1}$$

while the non-negativity and capacity constraints on the outflow can be expressed as

$$0 \leq z \leq c\,. \tag{2}$$

Traffic propagates among consecutive lanes according to a *routing matrix* $R \in \mathbb{R}_+^{n \times n}$ whose (i, j)-th entry R_{ij} — referred to as a *turning ratio*— represents the fraction of flow out of lane i that proceeds towards lane j. Conservation of mass implies that $\sum_{j \in \mathcal{E}} R_{ij} \leq 1$ for all $i \in \mathcal{E}$, the quantity $1 - \sum_{j \in \mathcal{E}} R_{ij} \geq 0$ representing the fraction of flow out of lane i that leaves the network directly from lane i. In other terms, the routing matrix R is row-substochastic. Inflows from the external environment are modeled by an exogenous and possibly time-varying *arrival vector* $\lambda = \lambda(t) \in \mathbb{R}_+^n$, whose entries $\lambda_i \geq 0$ describe the external inflows on the lanes $i \in \mathcal{E}$.

Definition 1. The routing matrix R is said to be: *adapted* to $\mathcal{G}$ if $R_{ij} = 0$ for all $i, j \in \mathcal{E}$ such that $\tau_i \neq \sigma_j$, i.e., $R_{ij} = 0$ whenever lane i does not end in the junction where lane j starts; *outflow-connected* if, for every $i \in \mathcal{E}$, there exists some $j \in \mathcal{E}$ with $\sum_{k \in \mathcal{E}} R_{jk} < 1$ and a path $i = i_0, i_1, \dots, i_l = j$ that starts in i, ends in j, and is such that $\Pi_{1 \leq i \leq l} R_{i-1,i} > 0$; *inflow-connected* with respect to an arrival vector $\lambda \in \mathbb{R}_+^n$ if, for every $j \in \mathcal{E}$, there exists some $i \in \mathcal{E}$ and a path $i = i_0, i_1, \dots, i_l = j$ that starts in i, ends in j, and is such that $\Pi_{1 \leq i \leq l} R_{i-1,i} > 0$.

For a given network topology $\mathcal{G}$, a routing matrix R adapted to $\mathcal{G}$, and an arrival vector λ, we consider the traffic network dynamics

$$\dot{x} = \lambda + (R^T - I) z\,. \tag{3}$$

Observe that the i-th row of equation (3),

$$\dot{x}_i = \lambda_i + \sum_j R_{ji} z_j - z_i\,,$$

can be interpreted as a law of mass conservation as it equates the traffic volume's growth rate $\dot{x}_i$ to the imbalance between the total inflow in lane i and the total outflow z_i from it, the former being given by the sum of the arrival rate and the total outflow from other lanes that is directed to lane i.

In addition to the capacity and non-negativity constraints (2), the outflow vector z is required to satisfy *scheduling* constraints as follows. Let a feasible *phase* be a subset of lanes that can be given green light simultaneously,

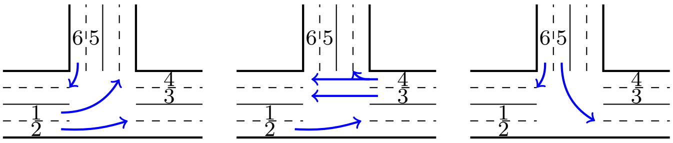

Fig. 1. Three non-zero phases for network with 6 lanes.

and let $\mathcal{P} \subseteq \{0,1\}^{\mathcal{E}}$ be the set of all feasible phases. We shall denote by $p = |\mathcal{P}|$ the total number of feasible phases and compactly represent the feasible phase set $\mathcal{P}$ as a binary matrix $P \in \{0,1\}^{n \times p}$ whose entries P_{ij} are such that $P_{ij} = 1$ if lane i is given green light during phase j, and $P_{ij} = 0$ otherwise. Throughout, we shall assume that the empty phase (green light to no lane) is always a feasible phase, equivalently, that the feasible phase matrix P contains a column of all 0s, that, without loss of generality we will assume being the last, i.e., the p-th, one. E.g., the network in Fig. 1 has $n = 6$ lanes and $p - 1 = 3$ non-zero feasible phases: its phase matrix is

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T .$$

Let us denote the unit p-simplex by

$$\mathcal{U} = \{u \in \mathbb{R}^p_+ : \mathbb{1}^T u = 1\}$$

and let

$$u \in \mathcal{U} \qquad (4)$$

be a control signal whose entries are to be interpreted as the fractions of time allocated to each phase. Considering that $0 \le z_i \le c_i$ when lane i is given green light whereas $z_i = 0$ when it is not, we have that, for a given control signal $u \in \mathcal{U}$ the outflow vector must satisfy the constraint

$$0 \le z \le CPu \,. \qquad (5)$$

Observe that (4)–(5) imply (2), but not *vice versa*, i.e., for any outflow $z \le c$, there may not exist a z satisfying (4)–(5) except for the trivial case when $\mathcal{P}$ contains the all-1 phase (green light to every lane simultaneously). Moreover, we will assume that the outflow from a nonempty lane is always the maximum possible given the control u, i.e., that

$$x'(CPu - z) = 0 \,. \qquad (6)$$

In fact, the constraint above, combined with (5) implies that the inequality $z_i \le c_i \sum_j (P_{ij} u_j)$ can be strict only when $x_i = 0$: indeed, allowing for the possibility of a strict inequality $z_i < c_i \sum_j (P_{ij} u_j)$ when $x_i = 0$ proves necessary in order to meet the nonnegativity constraint $x_i \ge 0$.

Throughout, we will use the following definition of solution of the traffic network dynamics and of its stability.

Definition 2. A *solution* of the traffic network dynamics associated to a routing matrix R adapted to a network topology $\mathcal{G}$ and a possibly time-varying arrival vector λ is a triple of trajectories $(x(t), z(t), u(t))_{t \ge 0}$ such that $x(t)$ is absolutely continuous and the constraints (1)–(6) are satisfied for almost all $t \ge 0$. A solution of the traffic network dynamics is *stable* if there exists a constant vector $b \in \mathbb{R}^n_+$ such that $x(t) \le b$ for all $t \ge 0$. The traffic network dynamics is said to be stable if all its solutions are stable.

Proposition 1. (Necessary condition for stability). Let R be an outflow-connected routing matrix adapted to a network topology $\mathcal{G}$ and λ a possibly time-varying arrival vector. Let P be a feasible phase matrix with p phases, $\mathcal{U}$ the unit p-simplex, and

$$\overline{CP\mathcal{U}} := \{z \in \mathbb{R}^n : 0 \le z \le CPu \text{ for some } u \in \mathcal{U}\} \,. \qquad (7)$$

If the traffic dynamics (1)–(6) admit a stable solution, then the average arrival vector $\overline{\lambda}(t) = \frac{1}{t}\int_0^t \lambda(s)\mathrm{d}s$ satisfies

$$\lim_{t \to +\infty} \operatorname{dist}\left((I - R^T)^{-1}\overline{\lambda}(t), \overline{CP\mathcal{U}}\right) = 0 \,. \qquad (8)$$

In particular, if the arrival vector $\lambda \in \mathbb{R}^n_+$ is constant, then

$$(I - R^T)^{-1}\lambda \in \overline{CP\mathcal{U}} \,. \qquad (9)$$

Proposition 1 establishes a fundamental limit for stability that depends only on the arrival rates, network topology, lane capacities, and phase set, but otherwise holds true for every control strategy (e.g., time-varying, feedback, feed-forward) and every solution of the traffic network dynamics (1)–(6). In particular, it does not have any implication on the existence and uniqueness of such solutions.

In fact, standard results from the theory of differential inclusions (Aubin and Cellina, 1984, Theorem 4, p. 101) guarantee that, if $u \in \mathcal{U}(x)$ where $x \mapsto \mathcal{U}(x) \subseteq \mathcal{U}$ is closed, convex and upper semicontinuous as a set-valued map, then existence (but not, in general, uniqueness) of continuous solutions is guaranteed. The following result establishes existence and uniqueness of a solution to the traffic network dynamics when using static Lipschitz-continuous feedback controls.

Theorem 1. (Existence and uniqueness of solutions). Let R be an outflow-connected routing matrix adapted to a network topology $\mathcal{G}$ and λ a possibly time-varying arrival vector. Let P be a feasible phase matrix with p phases, $\mathcal{U}$ be the unit p-simplex, and $x \mapsto u(x) \in \mathcal{U}$ be a static feedback control policy that is Lipschitz-continuous on $\mathbb{R}^n_+$. Then, for every nonnegative initial traffic volume $x(0)$, the traffic network dynamics (1)–(6) with feedback control $u = u(x)$ admit a unique solution.

The proof of Theorem 1 relies on a generalization of the reflection principle, Harrison and Reiman (1981), which has previously been applied to open loop traffic light control in Muralidharan et al. (2015), to cases with feedback.

3. DECENTRALIZED TRAFFIC SIGNAL CONTROLS AND PROPORTIONAL ALLOCATION POLICIES

In this section, we first introduce the notion of decentralized feedback controls, and then introduce the generalized proportional allocation policies. Let

$$\mathcal{E} = \bigcup_{1 \le k \le q} \mathcal{E}_k \,, \qquad \mathcal{E}_k \cap \mathcal{E}_{k'} = \emptyset \,, \quad k \ne k' \qquad (10)$$

be a partition of the set of lanes. We refer to such a partition (10) as *compatible* with the feasible phase set $\mathcal{P} \subseteq \{0,1\}^n$ if the latter can be written as the direct sum of the subsets of phases supported on each $\mathcal{E}_k$, i.e., if

$$\mathcal{P} = \bigoplus_{1 \le k \le q} \mathcal{P}_k \,, \qquad \mathcal{P}_k = \{\psi \in \mathcal{P} : \psi_i = 0 \;\; \forall i \in \mathcal{E} \setminus \mathcal{E}_k\} \,. \qquad (11)$$

Observe that, with this construction of $\mathcal{P}_k$, the all zero vector belongs to each $\mathcal{P}_k$. For $1 \le k \le q$, put $n_k = |\mathcal{E}_k|$, $p_k = |\mathcal{P}_k|$, and let the projection matrix on the n_k-dimensional subspace of vectors in $\mathbb{R}^n$ supported on $\mathcal{E}_k$ be denoted by $\Lambda^{(k)}$, so that

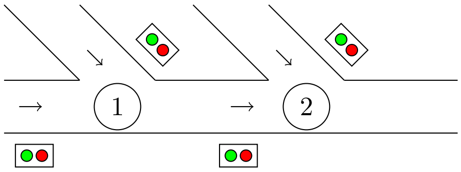

Fig. 2. The two merges in Example 1.

$$n=\sum_{k=1}^{q} n_k\,,\quad p=\prod_{k=1}^{q} p_k\,,\quad \psi=\sum_{1\le k\le q}\Lambda^{(k)}\psi\,,\quad \psi\in\mathcal{P}\,.$$

Then, the direct sum in (11) means that

$$\Lambda^{(k)}\psi\in\mathcal{P}_k\,,\qquad \psi\in\mathcal{P}\,,\qquad 1\le k\le q\,.$$

Observe that at least one trivial compatible partition always exists, with $q=1$, $\mathcal{E}_1=\mathcal{E}$, and $\mathcal{P}_1=\mathcal{P}$. A typical case of non-trivial partition of the lane set $\mathcal{E}$ that is compatible with $\mathcal{P}$ is obtained when phases are independent across different junctions: in this case, one can choose $q=m$ equal to number of nodes in the graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ and let $\mathcal{E}_k$ coincide with the set of in-links from each node $k\in\mathcal{V}$.

For a partition (10) of the lane set $\mathcal{E}$ that is compatible with the phase set $\mathcal{P}$, let

$$\mathcal{U}_k=\{u^{(k)}\in\mathbb{R}_+^{p_k}:\mathbb{1}^T u^{(k)}=1\}\,,\qquad 1\le k\le q$$

be the unit p_k-simplex and denote by $P^{(k)}\in\{0,1\}^{n\times p_k}$ the binary matrix whose columns coincide with the phases in $\mathcal{P}_k$. It follows that, for every control signal $u\in\mathcal{U}$, where $\mathcal{U}$ is the unit p-simplex, one has that

$$Pu=\sum_{1\le k\le q}P^{(k)}u^{(k)}\,,\qquad u^{(k)}\in\mathcal{U}_k\,,\qquad 1\le k\le q\,.$$

In other terms, there is no loss of generality in restricting attention to control signals $u\in\mathcal{U}$ of the form

$$u_j=\prod_{1\le k\le q}u^{(k)}_{h_k(j)}\,,\qquad 1\le j\le p\,,\qquad (12)$$

where $u^{(k)}\in\mathcal{U}_k$ and $1\le h_k(j)\le p_k$ is the index such that

$$P_{ij}=(\Lambda^{(k)}P)_{i,(h_k(j))}\,,\qquad 1\le i\le n\,,\qquad 1\le j\le p\,.$$

Example 1. Consider a small network with two consecutive merges, as depicted in Fig. 2. Suppose that each merge is controlled separately, and each merge can only give green light to one incoming lane at the time. Then

$$P^{(1)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}^T\quad\text{and}\quad P^{(2)}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}^T\,.$$

Hence, P is given by

$$P=\begin{bmatrix}1&0&0&0&1&0&1&0&0\\0&1&0&0&0&1&0&1&0\\0&0&1&0&1&1&0&0&0\\0&0&0&1&0&0&1&1&0\end{bmatrix}\,.$$

Now, suppose that $u^{(1)}=[0.2\ 0.3\ 0.5]^T$ and $u^{(2)}=[0.1\ 0.7\ 0.2]^T$. Then, e.g., u_3 is given by

$$u_3=u^{(1)}_{h_1(3)}\cdot u^{(2)}_{h_2(3)}=u^{(1)}_3\cdot u^{(2)}_1=0.5\cdot 0.1=0.05\,,$$

and the full u-vector is given by

$$u=[0.02\ 0.03\ 0.05\ 0.14\ 0.04\ 0.21\ 0.35\ 0.06\ 0.1]^T\,.$$

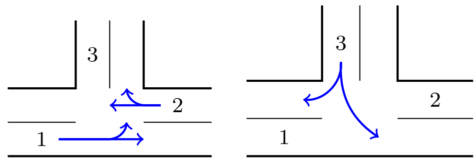

Fig. 3. An example of orthogonal phases. Here lane 1 and 2 belong to one phase, and lane 3 to another.

We will refer to feedback controls $u(x)$ as *decentralized* according to a compatible partition (10) if

$$u_j(x)=\prod_{1\le k\le q}u^{(k)}_{h_k(j)}(x^{(k)})\,,\qquad 1\le j\le p\,,\qquad (13)$$

where

$$x^{(k)}=\Lambda^{(k)}x\,,\qquad 1\le k\le q\,,$$

is the vector of local state information. Notice that, for non-trivial compatible partitions, in contrast to (12), equation (13) imposes an actual restriction, as it constrains the local control signal $u^{(k)}$ on depending on local state feedback $x^{(k)}$ only, as opposed to global state feedback x.

Let $\mathcal{P}\subseteq\{0,1\}^n$ be a set of p feasible phases containing the empty phase 0, and $P\in\{0,1\}^{n\times p}$ be the corresponding phase matrix. We refer to $\mathcal{P}$ as an *orthogonal* feasible phase set if all phases are disjoint, equivalently, if the every pair of columns of P has null scalar product. For an example of orthogonal phases, see Fig. 3. Throughout this section we shall focus on orthogonal phase sets, while we shall generalize our results to possibly nonorthogonal phase sets in Section 5.

Given an orthogonal set of admissible phases $\mathcal{P}$, a compatible partition of the set of lanes as in (10), and a vector $\xi\in\mathbb{R}^q$ with strictly positive entries, define the *generalized proportional allocation* control as the decentralized feedback control (13) with, for $1\le k\le q$,

$$\begin{aligned}u^{(k)}_{p_k}(x^{(k)})&=\frac{\xi_k}{\zeta_k(x)}\,,\quad \zeta_k(x)=\xi_k+(x^{(k)})^T P^{(k)}\mathbb{1}\,,\\ u^{(k)}_h(x^{(k)})&=\frac{1}{\zeta_k(x)}\sum_{j\in\mathcal{E}_k}P^{(k)}_{jh}x_j\,,\qquad 1\le h<p_k\,.\end{aligned}\qquad (14)$$

The positive parameters ξ_k can be interpreted as capturing in our continuous-time model the constraint that a fraction of the cycle length is allocated to the all-red phase (i.e., phase index p_k) as required by, e.g., safety concerns and phase changes. Notice that their introduction makes the feedback controls in (14) Lipschitz-continuous in x, hence Theorem 1 can be applied in order to establish existence and uniqueness of a solution for every initial traffic volume vector $x(0)$. The reason for referring to the decentralized feedback control (13)–(14) as generalized proportional allocation control is clarified by the following special case.

Example 2. For a partition of the set of lanes as in (10), let the feasible phase set be

$$\mathcal{P}=\bigoplus_{1\le k\le q}\mathcal{P}_k\,,\quad \mathcal{P}_k=\{0\}\cup\{\delta^{(i)}:i\in\mathcal{E}_k\}\,,\quad 1\le k\le q\,.$$

I.e., the feasible phases are those whereby at most one lane from each subset $\mathcal{E}_k$ can be activated simultaneously. Let us label lanes so that $\mathcal{E}_k=\{i_{k,h}:1\le h\le n_k\}$ and observe that $p_k=n_k+1$ for $1\le k\le q$. We can then order columns in $P^{(k)}$ in such a way that the all-zero one comes last (with index $n_k + 1$) while, for $1 \le h \le n_k$ the h-th column of $P^{(k)}$ has a 1 in its $i_{k,h}$-th entry and all zeros elsewhere. Then, (14) reduces to

$$u_h^{(k)}(x^{(k)}) = \frac{x_{i_{k,h}}}{\xi_k + \sum_{1\le l\le n_k} x_{i_{k,l}}}\,, \quad 1 \le h \le n_k\,,$$
$$u_{n_k+1}^{(k)}(x^{(k)}) = \frac{\xi_k}{\xi_k + \sum_{1\le l\le n_k} x_{i_{k,l}}}\,,$$

that shows that priority is allocated to the different lanes in each $\mathcal{E}_k$ proportionally to their current traffic volume.

4. STABILITY ANALYSIS

We will assume from now on that the exogenous arrival are constant s.t. $\lambda(t) = \lambda$, then the arrival rate for each lane at equilibrium, $a \in \mathbb{R}_+^{\mathcal{E}}$, can be computed by

$$a = (I - R^T)^{-1}\lambda\,.$$

We will moreover assume that the routing matrix R is *inflow-connected* with respect to λ, so that $a > 0$.

Theorem 2. (Stability of proportional allocation policies). Let R be a routing matrix adapted to a network topology $\mathcal{G}$ and λ a constant arrival vector, such that R is both outflow-connected and inflow-connected with respect to λ. Let $\mathcal{P}$ be a feasible phase set with p phases and corresponding matrix P, $\mathcal{U}$ the unit p-simplex, and $\widetilde{CP\mathcal{U}}$ be the interior of $CP\mathcal{U}$. For any partition (10) of the lane set that is compatible with $\mathcal{P}$, let $u(x)$ be the proportional controller given by (13)–(14). Then, if

$$(I - R^T)^{-1}\lambda \in \widetilde{CP\mathcal{U}}\,, \tag{15}$$

the traffic network dynamics (1)–(6) are stable and every solution $x(t)$ approaches the set

$$\mathcal{X} = \{x \in \mathbb{R}_+^{\mathcal{E}} : x^T \left(CPu(x) - (I - R^T)^{-1}\lambda\right) = 0\}$$

as t grows large.

Remark 1. In the specific case in Example 2 the solution to the dynamics (1)–(3) converges to a globally asymptotically stable equilibrium $x^* \in \mathbb{R}_+^{\mathcal{E}}$, which was proven in Nilsson et al. (2015).

5. PROPORTIONAL ALLOCATION CONTROL WITH NONORTHOGONAL PHASES

In this section, we discuss extensions of the control policy and stability results of Sections 3 and 4, respectively, to the case when phases are not necessarily orthogonal. We start by observing that, in the case of orthogonal phases, the controller in (14) coincides with the unique solution to the following concave maximization problem

$$u^{(k)}(x) \in \underset{\nu\in\mathcal{U}_k}{\operatorname{argmax}} \sum_{i\in\mathcal{E}_k} x_i \log \sum_{1\le j<p_k} P_{ij}^{(k)}\nu_j + \xi_k \log \nu_{p_k}\,. \tag{16}$$

For the non-orthogonal phase sets, we define the control policy as any choice of an optimal solution in the maximization problem (16). In this case, the control signal may not be uniquely determined, as the following example shows.

Example 3. Consider a partition k with three lanes (indexed $\{1, 2, 3\}$), all with unit capacity. Let the phase matrix be

$$P^{(k)} = \begin{bmatrix}1&0&0\\1&1&0\\0&1&0\end{bmatrix}.$$

The maximization problem in (16) then becomes

$$u^{(k)}(x) \in \underset{\nu\in\mathcal{U}_k}{\operatorname{argmax}} \quad x_1\log(\nu_1) + x_2\log(\nu_1+\nu_2)$$
$$x_3\log(\nu_2) + \xi_k\log(\nu_3)\,.$$

The solution to the maximization problem is:

- If $x_1 = 0, x_2 > 0, x_3 = 0$, then
$$0 \le u_1 \le \frac{x_2}{x_2+\xi_k}\,, \quad u_2 = \frac{x_2}{x_2+\xi_k} - u_1\,,$$
$$u_3 = 1 - u_1 - u_2 = 1 - \frac{x_2}{x_2+\xi_k}\,.$$
- For all other cases,
$$u_1 = \frac{x_1(x_1+x_2+x_3)}{(x_1+x_3)(x_1+x_2+x_3+\xi_k)}\,, \quad u_2 = \frac{x_3}{x_1}u_1\,.$$

Let us specifically point out the need of differential inclusion in our model. Assume that the lanes have exogenous inflows, λ_1, λ_2 and λ_3, respectively, and no inflows from other lanes. Let $x_1 = 0$ and $x_3 = 0$, then

$$u_1 + u_2 = \frac{x_2}{x_2+\xi_k}\,.$$

Now suppose that $\lambda_1 + \lambda_3 < u_1 + u_2$. To keep $x(t) \ge 0$, we have to choose z_1, z_2 such that $z_1 \le \lambda_1$ and $z_3 \le \lambda_3$. However, choosing $z_1 < \lambda_1$ or $z_3 < \lambda_3$, will make $\dot{x}_1 > 0$ or $\dot{x}_3 > 0$, and the traffic volumes will become positive. Let us for simplicity assume that $z_1 = 0$ and $z_3 = \lambda_3$, then after a sufficiently small time, $x_1 > 0$ and

$$u_1 = \frac{x_1+x_2}{x_1+x_2+\xi_k} > \lambda_1,$$

and x_1 will immediately go back to zero again. Therefore this solution can not be absolutely continuous. To get an absolutely continuous solution in this case one has to choose $z_1 = \lambda_1$ and $z_3 = \lambda_3$.

Remark 2. From Example 3 it is easy to observe that the equilibrium does not have to be unique. It follows that if $a_2 > a_1 + \lambda_3$ the equilibrium will be $x_1^* = 0$, $x_2^* > 0$ and $x_3^* = 0$. On the other hand, if $\lambda_2 < \lambda_1 + \lambda_3$ the equilibrium will instead be $x_1^* > 0$, $x_2^* = 0$ and $x_3^* = 0$. When $\lambda_2 = \lambda_1 + \lambda_3$ the equilibrium will depend on the initial state, since there exists many possible choices of $x_1 > 0, x_2 > 0, x_3 > 0$ such that

$$a_1 = u_1 = \frac{x_1(x_1+x_2+x_3)}{(x_1+x_3)(x_1+x_2+x_3+\xi_k)}\,,$$
$$a_3 = u_2 = \frac{x_3(x_1+x_2+x_3)}{(x_1+x_3)(x_1+x_2+x_3+\xi_k)}\,.$$

Even if the control signal is not Lipschitz anymore, it follows from the Maximum Theorem, see (Sundaram, 1996, Theorem 9.14), that $u(x)$ is upper semi-continuos and convex-valued, since the objective function in the optimization problem (16) is a concave function in ν. Hence existence of solutions to the dynamics (1)–(6) together with the controller (16) can still be ensured, while uniqueness is still an open problem. Note that, when $x_i > 0$, $h_i(x)$ is uniquely determined. Hence the proof of Theorem 2 works for non-orthogonal phases as well.

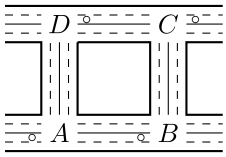

Fig. 4. The four intersections used for simulations.

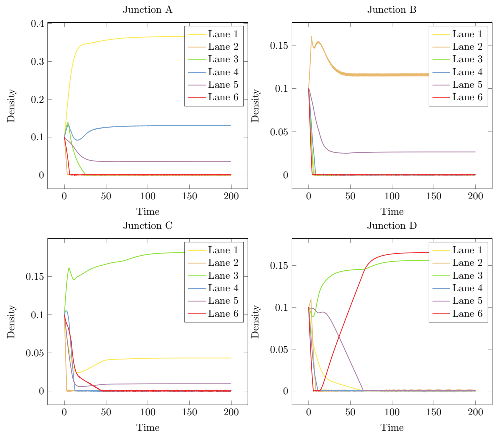

Fig. 5. How the traffic volumes evolves with time in the simulation.

Corollary 1. The stability results stated in Theorem 2 holds for all control signals determined by (16) when $\mathcal{P}$ is a feasible set of phases.

6. NUMERICAL SIMULATIONS

In this section we report numerical simulations of the continuous-time dynamics given by (1)–(6) together with the controller given by (16) for a small network with with four intersections as shown in Fig. 4. For each intersection, the phases are the same as in Fig. 1, where the position of the 1-lane from Fig. 1 is marked by a circle, $\circ$, in Fig. 4. In Fig. 5 it is shown how the traffic volume on each line evolves with time, when all lanes start with the initial traffic volume $x_i(0) = 0.1$.

7. CONCLUSIONS

In this paper we have presented a feedback-based traffic signal control policy that only requires information about the traffic volume in order to stabilize network. We have also showed that the proposed policy is maximally stabilizing, i.e., when any controller can stabilize the network, the proposed one is able to stabilize as well. Currently pursued research directions include a comparison with the back-pressure controller in a micro simulator, the investigation of finite storage capacities, and studying robustness of the controller with respect to the traffic propagation model.

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