Generalized Proportional Allocation Policies for Robust Control of Dynamical Flow Networks

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***Abstract*—We study a robust control problem for dynamical flow networks. In the considered framework, traffic flows along the links of a transportation network —modeled as a capacited multigraph— and queues up at the nodes, whereby control policies determine which incoming queues are to be allocated service simultaneously, within some predetermined scheduling constraints. We first prove fundamental performance limitations on the system performance by showing that for a dynamical flow network to be stabilizable by some control policy it is necessary that the exogenous inflows belong to a certain stability region, that is determined by the network topology, the link capacities, and the scheduling constraints. Then, we introduce a family of distributed controls, referred to as Generalized Proportional Allocation (GPA) policies, and prove that they stabilize a dynamical transportation network whenever the exogenous inflows belong to such stability region. The proposed GPA control policies are decentralized and fully scalable as they rely on local feedback information only. Differently from previously studied maximally stabilizing control strategies, the GPA control policies do not require any global information about the network topology, the exogenous inflows, or the routing, which makes them robust to unpredicted network load variations and changes in the link capacities or the routing decisions. Moreover, the proposed GPA control policies also take into account the overhead time while switching between services. Our theoretical results find one application in the control of urban traffic networks with signalized intersections, where vehicles have to queue up at junctions and the traffic signal controls determine the green light allocation to the different incoming lanes.**

Index terms: Dynamical flow networks, transportation networks, robust control, distributed control, non-linear control, traffic signal control.

I. Introduction

Dynamical flow networks have attracted significant research interest, with applications including road traffic, data, production, and biological networks [2]–[6]. Such network systems tend to be of large scale, involve complex interactions between different layers, and are potentially fragile to cascading failures [7]–[9]. In order to deal with such complexity, the role of structural properties such as monotonicity, contractivity, separable Lyapunov functions, and convexity has proved critical in order to design scalable distributed control architectures with provable performance and robustness guarantees [10]–[19].

In this paper, we study a robust control problem for dynamical flow networks modeled as deterministic continuous-time point-queue networks. In the considered framework, traffic flows along the links of a capacited multigraph modeling the transportation network, and queues up at the nodes, while satisfying the mass conservation law. At each node, scheduling control policies determine which incoming queues are to be allocated service simultaneously. We study the problem in the case where not all incoming queues at a node can receive service simultaneously as there are scheduling constraints modeled in terms of phases and the service allocation to such different phases is determined by the controller.

This paper's main contribution consists in the introduction of a family of distributed controls, referred to as Generalized Proportional Allocation (GPA) policies and in the proof of their throughput optimality. Albeit relying only on local feedback information on the queue lengths on the incoming links to a node —which makes the GPA control policies fully decentralized and scalable with the network size— and requiring no global information on the network topology, nor on the exogenous inflows, nor on the routing, we prove that the proposed GPA control policies are maximally stabilizing. In particular, we show that they are able to stabilize a dynamical flow network with given topology, scheduling constraints, exogenous inflows, and routing, whenever any controller can. These properties make the GPA control policies robust to perturbations to the exogenous inflows and model errors regarding the routing of the traffic. Recently, we have proved its robustness to measurement offsets as well [20].

Apart from being a natural model for deterministic point-queues, the dynamical flow network models studied in this paper are also related to the fluid limit approximations of stochastic queueing networks for which different service allocation controllers have been studied, see, e.g., [21], [22]. In particular, the BackPressure controller, first proposed in [21], determines both the service allocation and the routing. While this kind of control strategy can be applied in some scenarios, like communication networks, there are other applications where one cannot assume that it is the same controller that both determines the service allocation and the routing. In this paper, we focus on the problem where the routing is exogenous and only the service allocation can be directly controlled. For instance, in traffic signal control of urban transportation networks, this means that the drivers determine their routes themselves, while the only control action is how to allocate green light at signalized junctions.

While the main contribution of this paper is a theoretical

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This research was carried on within the framework of the MIUR-funded *Progetto di Eccellenza* of the *Dipartimento di Scienze Matematiche G.L. Lagrange*, Politecnico di Torino, CUP: E11G18000350001. It received partial support from the MIUR Research Project PRIN 2017 "Advanced Network Control of Future Smart Grids" (http://vectors.dieti.unina.it), the Swedish Research Council [2015-04066], and by the *Compagnia di San Paolo*.

Part of the results appeared in a preliminary form in [1].

analysis of a novel distributed control policy for dynamical flow networks, the presented mathematical abstraction can be viewed as a simplified model for transportation networks where the GPA controller controls the traffic signals. The large body of literature on traffic signal control starts with early works, e.g., [23] considering a centralized open-loop approach to coordinate the cycles in the traffic signals, so that they allow traffic on the main corridors in a city to progress smoothly, sometimes referred to as "green-waves". One early computer implementation of an algorithm that computes an optimal traffic signal control is TRANSYT [24], which computes a static signal program. Later, other approaches to compute the optimal offset in signal timing have been developed, for example in [25]–[27]. By utilizing magnetic loop detectors to detect vehicles, several solutions have been proposed on how to retune the traffic signal programs depending on the current state of the network. SCAT [28], SCOOT [29], UTOPIA [30] are all examples of such solutions. While those retuning strategies take several practical aspects into account, they do not have any formal performance guarantees, such as stability of the dynamical system or throughput optimality.

On the other hand, with the rapid development of new sensors such as, e.g., cameras, it is now possible to control traffic signals in real time. One recently proposed distributed feedback solution for traffic signal control is the MaxPressure controller, see [31]. In particular, the MaxPressure controller is based on the same idea as BackPressure [21], namely minimizing the drift of a separable Lyapunov function. However, differently from the BackPressure controller, the MaxPressure controller is only concerned with service allocation and not with routing. In fact, in order to minimize the drift of the Lyapunov function, the MaxPressure controller needs information about the vehicles' routing behaviors, something that is often difficult to get an exact estimate of, although estimation techniques have been proposed in, e.g., [32]. Under the assumption that the turning ratios of each junction are known, other feedback policies for traffic signals have been proposed based, e.g., on model predictive control [33]–[35]. Also, the idea of utilizing the routing suggestions from the BackPressure controller and variants thereof to control the vehicles' paths has been proposed in [36]–[38]. While the aforementioned works assume that the routing is exogenous, analysis of traffic controllers together with optimal route choices has been done in [39], [40].

Control policies relying on information about the routing may turn out to be less robust to perturbations. For example, today many drivers use online route guidance, something that makes it more likely that they will change their preferred routes from a trip to another. In contrast, our proposed GPA control policies do not require any information about the routing, and still are —just like the MaxPressure-controller— provably able to stabilize the dynamical flow network whenever any control strategy is able to do so. The particular structure of the GPA control policies —i.e., using only local feedback information on the queue lengths and not relying on any global knowledge of the network structure, the exogenous inflows or the routing— makes them easy to implement and robust to demand variations and unpredicted changes in the link capacities or the routing decisions. The intuition behind such GPA controls is related to the idea of proportional fairness, originally proposed for queueing networks, see, e.g., [22] and [41]. Our proof of maximal stability relies on a Lyapunov-LaSalle argument based on a particular separable Lyapunov function. Differently from previously proposed proportional allocation controllers, we also take into account the fact that in many service allocation tasks, a fraction of the service time can not be fully utilized when shifting between different service modes. In transportation networks, this is known as clearance time, and is the time when traffic signals are showing yellow light [42], while in CPU-scheduling this time to shift between different service allocations is referred to as a context switch [43]. Given that the overhead time is fixed, the length of the service cycle will vary. As we will show later in the paper, by taking this switching time into account, our controller will in some settings adjust the cycle length after the demand in the same way as Webster's formula [44] suggests. While this paper focuses on the mathematical aspects of the GPA controller for a simplified traffic flow model, i.e., a point queue model that does not capture propagation delays and congestion effects, simulation studies in [45], [46] of the GPA control policies in a traffic micro-simulator have validated our theoretical findings showing that by incorporating this overhead time in the controller, and adjust the cycle lengths accordingly, gives a better performance compared to standard proportional fairness control. These simulation studies have also shown that since the controller is decentralized, it can be employed in a city-wide transportation network, and still do all the computations in real-time.

The rest of this paper is organized as follows: The last paragraph of this section is devoted to introducing some notational conventions used throughout the paper. In Section II we present the dynamical flow network model. In Section III, we present a fundamental limit on how large exogenous inflows a flow network can possibly handle and still keep it stable. In Section V, we show simulations of the dynamics on a small flow network, that also illustrate the controller's ability to adopt a new behavioral flow pattern, something that a static controller is not able to. The paper is concluded by discussing some ongoing and future research lines. Finally, the Appendix reports proofs of all the technical lemmas in the paper.

Notation Let: $\mathbb{R}_{(+)}$ denote the (non-negative) reals. For a finite set $\mathcal{A}$, let $\mathbb{R}^{\mathcal{A}}$ denote the set of vectors indexed by the elements in $\mathcal{A}$. For a vector $a \in \mathbb{R}^n$, let $\mathrm{diag}(a) \in \mathbb{R}^{n \times n}$ be a diagonal matrix with the components of a on its diagonal. With $\mathbb{1}$ we denote a vector whose all elements equals one. The positive and negative parts of a vector x are denoted by $[x]_+ = \max(x, 0)$ and $[x]_- = \max(-x, 0)$, respectively, where $\max$ and $\min$ are applied entry-wise. We use $\|x\|$ to denote the Euclidean norm of a vector x, unless otherwise specified. For a subset $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we let $\mathrm{dist}(x, \mathcal{A})$ denote the shortest distance to the set, i.e., $\mathrm{dist}(x, \mathcal{A}) = \inf_{a \in \mathcal{A}} \|x - a\|$. For finitely many sets $\mathcal{A}_1, \mathcal{A}_2 \ldots, \mathcal{A}_n$, $\Pi_{k=1}^n \mathcal{A}_k$ denotes their cartesian product set.

II. Dynamical flow network model

In this section, we describe the dynamical flow network model in detail and formulate the associated control problem. The section is concluded with some examples illustrating the introduced modeling concepts.

The topology of the flow network is described as a capacited directed multigraph $\mathcal{G}=(\mathcal{V},\mathcal{E},c)$. Here, $\mathcal{V}$ and $\mathcal{E}$ denote the finite sets of nodes and directed links, respectively, whereas c in $\mathbb{R}_+^{\mathcal{E}}$ is a vector whose entries $c_i>0$ represent the flow capacities of the different links i in $\mathcal{E}$. We shall denote the number of nodes by $|\mathcal{V}|=m$ and the number of directed links by $|\mathcal{E}|=n$. For simplicity, we may identify $\mathcal{V}=\{v_1,\dots,v_m\}$ and $\mathcal{E}=\{1,\dots,n\}$. Each link i in $\mathcal{E}$ is directed from its *tail* node σ_i to its *head* node τ_i. We shall assume that $\sigma_i\neq\tau_i$ for every link i in $\mathcal{E}$, i.e., that $\mathcal{G}$ does not contain any self-loop. On the other hand, letting $\mathcal{G}$ be a multigraph rather than simply a graph allows for the possibility of parallel links between two nodes, i.e., links with the same tail and head nodes. A length-l walk in $\mathcal{G}$ is an l-tuple of links $(e_1,\dots,e_l)$ in $\mathcal{E}^l$ such that the tail node of the next link coincides with the head node of the previous link, i.e., $\tau_{e_{h-1}}=\sigma_{e_h}$ for every $1\leq h\leq l$. A length-l path in $\mathcal{G}$ is a walk $(e_1,\dots,e_l)$ that does not pass through the same node twice, i.e., such that $v_0=\sigma_{e_1}$ and $v_h=\tau_{e_h}$ for $1\leq h\leq l$ satisfy $v_r\neq v_s$ for $0\leq r<s\leq l$, except possibly for $v_0=v_l$, in which case the path is referred to a *cycle*.

We identify the directed links i in $\mathcal{E}$ with *cells*. Traffic flows from a cell i to cells j that are immediately downstream of i, i.e., such that $\tau_i=\sigma_j$. The *traffic volume* in and the *outflow* from a cell i are denoted by x_i and z_i, respectively, and they are both nonnegative quantities. Moreover, we assume that the outflow z_i from a cell i never exceeds the link flow capacity. Such non-negativity and capacity constraints hence read

$$x_i\geq 0\,,\qquad 0\leq z_i\leq c_i\,,\qquad i\in\mathcal{E}\,. \tag{1}$$

Cells i in $\mathcal{E}$ may get an exogenous traffic inflow $\lambda_i\geq 0$ from outside the network. Traffic volumes, outflows, and exogenous inflows are in general time-varying; when useful we shall emphasize their time dependence by writing $x_i(t)$, $z_i(t)$, and $\lambda_i(t)$, respectively. The vectors of all cells' traffic volumes, outflows, and exogenous inflows are denoted by x, z, and λ, respectively. We shall also use the compact notation $\mathcal{X}=\mathbb{R}_+^{\mathcal{E}}$ for the state space of the network flow dynamics, and write

$$C=\operatorname{diag}(c)$$

for the diagonal matrix of the cells' flow capacities.

To model flow propagation through the network, we introduce a *routing matrix* R in $\mathbb{R}_+^{\mathcal{E}\times\mathcal{E}}$ whose entries R_{ij} are all nonnegative and represent the fraction of the outflow from cell i to a downstream cell j. Topological constraints imply that $R_{ij}=0$ whenever $\tau_i\neq\sigma_j$, i.e., if cell j is not immediately downstream of cell i. On the other hand, conservation of mass implies that $\sum_j R_{ij}\leq 1$ for every cell i in $\mathcal{E}$, a constraint that can be compactly rewritten as $R\mathbb{1}\leq\mathbb{1}$. If $\sum_j R_{ij}<1$ for a cell i, this means that the fraction $1-\sum_j R_{ij}>0$ of the outflow z_i leaves the network when flowing out from cell i. Otherwise, if $\sum_j R_{ij}=1$, this means that no traffic flows out of the network directly from cell i, so that all the outflow from cell i is distributed among its immediately downstream cells.

A cell j is said to be *reachable* from a cell i through a routing matrix R if $i=j$ or there exists a path $(e_1,\dots,e_l)$ such that $e_1=i$, $e_l=j$, and $\Pi_{1\leq h<l}R_{e_h,e_{h+1}}>0$. The pair of an exogenous inflow vector λ and a routing matrix R is said to be *out-connected* if for every cell i with $\lambda_i>0$ there exists a cell j that is reachable from i through R and such that $\sum_{k\in\mathcal{E}}R_{jk}<1$. In the same manner, a pair (λ,R) is said to be *in-connected* if for every cell j there exists some cell i with $\lambda_i>0$ such that j is reachable from i through R. The routing matrix R is then referred to as out-connected if (λ,R) is out-connected for every inflow vector λ in $\mathbb{R}_+^n$, i.e. if from every cell i a cell j with $\sum_{k\in\mathcal{E}}R_{jk}<1$ is reachable. Finally, a routing matrix R is in-connected if (λ,R) is in-connected for every λ in $\mathbb{R}_+^n\setminus\{0\}$, i.e, if every cell j is reachable from every other cell i.

The traffic flow dynamics on a flow network with topology $\mathcal{G}=(\mathcal{V},\mathcal{E},c)$ then reads

$$\dot{x}_i=\lambda_i+\sum_{j\in\mathcal{E}}R_{ji}z_j-z_i\,,\qquad\forall i\in\mathcal{E}\,. \tag{2}$$

In addition to the non-negativity and capacity constraints (1), the flow network is characterized by scheduling constraints on which traffic can simultaneously flow from a cell i to an immediately downstream one j through node $k=\tau_i=\sigma_j$. A cell is said to be *served* if traffic is allowed to flow out from it. In order to describe the scheduling constraints, we introduce the notion of *phases* as follows. For every node k in $\mathcal{V}$, let $\mathcal{E}_k=\{i\in\mathcal{E}\mid\tau_i=k\}$ be the set of incoming cells and let $n_k=|\mathcal{E}_k|$ be its cardinality. A *local phase* at node k is then a subset $\mathcal{Q}\subseteq\mathcal{E}_k$ of incoming cells that can be served simultaneously. Let $\mathcal{P}_k$ be the set of feasible local phases and $p_k=|\mathcal{P}_k|$ be its cardinality. Such set of feasible local phases at a node k can be represented in terms of a local phase matrix, that is a binary $n_k\times p_k$ matrix

$$P^{(k)}\in\{0,1\}^{\mathcal{E}_k\times\mathcal{P}_k}$$

whose entries are defined as

$$P^{(k)}_{ij}=\begin{cases}1 & \text{if cell } i\in\mathcal{E}_k \text{ is served in phase } j\in\mathcal{P}_k,\\ 0 & \text{if cell } i\in\mathcal{E}_k \text{ is not served in phase } j\in\mathcal{P}_k.\end{cases}$$

We then stack local phase matrices into a block-diagonal global phase matrix

$$P=\begin{bmatrix}P^{(v_1)} & & & \\ & P^{(v_2)} & & \\ & & \ddots & \\ & & & P^{(v_m)}\end{bmatrix}.$$

Without loss of generality, we assume throughout the paper that every cell belongs to at least one phase j in $\mathcal{P}=\bigcup_{k\in\mathcal{V}}\mathcal{P}_k$, i.e., $\sum_{j\in\mathcal{P}}P_{ij}\geq 1$ for every cell i, which we may rewrite more compactly as $P\mathbb{1}\geq\mathbb{1}$. Moreover, we shall refer to phases as *orthogonal* if every cell i in $\mathcal{P}_k$ belongs to exactly one local phase in $\mathcal{P}_k$, i.e., if $\sum_{j\in\mathcal{P}}P_{ij}=1$ for every cell i in $\mathcal{E}$, which we may rewrite more compactly as $P\mathbb{1}=\mathbb{1}$.

Remark 1: While this paper consider phases over the nodes, our results apply for arbitrary partitions of the cell set $\mathcal{E}$, i.e.,

$$\mathcal{E} = \bigcup_{k \in \mathcal{V}} \mathcal{E}_k \,, \qquad \mathcal{E}_k \cap \mathcal{E}_h = \emptyset \,, \ \forall h \neq k \in \mathcal{V} \,,$$

where $\mathcal{V}$ is a finite set of cardinality m.

Depending on the application, the phases can correspond to different kinds of actuators that can be activated simultaneously. For example, in traffic signal control of urban networks, the phases can be seen as lanes that can receive green light simultaneously in such a way that collisions are avoided.

From now on, we shall identify a *flow network* as the pair $(\mathcal{G}, P)$ of a topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, c)$ and a phase matrix P. To determine which phase should be activated at each node, we introduce the set of control signals

$$\mathcal{U} = \prod_{k \in \mathcal{V}} \mathcal{U}_k \,,$$

defined as the Cartesian product of the sets of local control signals

$$\mathcal{U}_k = \left\{ u \in \mathbb{R}_+^{\mathcal{P}_k} \mid \mathbb{1}^\top u \leq 1 \right\} , \qquad k \in \mathcal{V} \,.$$

The j-th entry u_j of a control signal u in $\mathcal{U}$ represents the fraction of time allocated to phase j. This definition of local control set $\mathcal{U}_k$ captures the fact that the total fraction of time $\sum_{i \in \mathcal{E}_k} u_i$ allocated to all local phases p in $\mathcal{P}_k$ at each node k in $\mathcal{V}$ must not exceed 1.

We shall allow for set-valued control signals that, at each time $t \geq 0$, determine a set $\mathcal{W}(t) \subseteq \mathcal{U}$ of controls that can be activated. The opportunity to allow for set-valued control signals will become apparent in the following. Phase control signals introduce constraints on the outflow vector $z(t)$ at time t that are generally more stringent than the flow capacity ones. Specifically, we have that

$$u(t) \in \mathcal{W}(t) \,, \qquad z_i(t) \leq c_i \sum_j P_{ij} u_j(t) \,, \quad \forall i \in \mathcal{E} \,. \quad (3)$$

The inequality above states that the outflow from a given cell i in $\mathcal{E}$ cannot exceed the capacity of cell i times the total fraction of time allocated by the control $u(t)$ in $\mathcal{W}(t)$ to all local phases in $\mathcal{P}_{\tau_i}$ containing cell i. While the above is an inequality, we shall in fact assume that it holds true as an equality whenever the traffic volume $x_i(i)$ is strictly positive. Using (1) and (3), this additional constraint can be written as

$$x_i(t) \left(c_i \sum_j P_{ij} u_j(t) - z_i(t) \right) = 0, \quad \forall i \in \mathcal{E} \,. \quad (4)$$

Observe that the dynamical flow network (1)–(4) is completely specified by the flow network $(\mathcal{G}, P)$, the exogenous inflow vector λ, the routing matrix R, and the control signal $(\mathcal{W}(t))_{t \geq 0}$. In this paper, we are particularly interested in investigating the case when the control set $\mathcal{W}(t)$ is determined by the current state of the network, so that

$$\mathcal{W}(t) = \omega(x(t)) \,, \qquad t \geq 0 \,,$$

where the *feedback control policy*

$$\omega : \mathcal{X} \ni x \mapsto \omega(x) \subseteq \mathcal{U} \,,$$

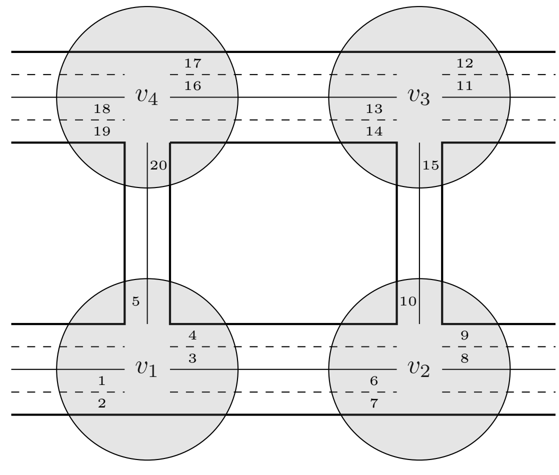

Fig. 1. Part of a transportation network consisting of four junctions, each of which corresponds to a node: the phases represent constraints on which lanes that can receive green light simultaneously.

is defined as a map from the state space $\mathcal{X}$ to the class of subsets of the control space $\mathcal{U}$.

For convenience of the notation, we introduce

$$\zeta(x) = CPv \,, \quad v \in \omega(x(t)) \,. \quad (5)$$

With the feedback control policy, the network flow dynamics (1)–(4) can then be compactly rewritten as

$$\dot{x} = \lambda - (I - R^\top) z \,, \quad (6)$$

with the constraints

$$x \geq 0 \,, \qquad 0 \leq z \leq \zeta(x) \,, \qquad x^\top (\zeta(x) - z) = 0 \,. \quad (7)$$

Equations (5)–(7) model the network traffic flow dynamics as a differential inclusion. We refer to a triple $(x(t), u(t), z(t))_{t \geq 0}$ as a solution of the controlled traffic flow dynamics if $x(t)$ is an absolutely continuous function of t, $u(t)$, and $z(t)$ are measurable functions of t, and (5)–(7) are satisfied.

In this paper, we shall not discuss issues of existence and uniqueness of solutions of (5)–(7), as the presented results will hold true for any solution (provided it exists and regardless whether it is unique or not). The interested reader is referred to, e.g., [47] where the existence and the uniqueness of a solution of (5)–(7) are proved in the case when the control policy $\omega(x)$ is single-valued and Lipschitz continuous with respect to x.

A. Examples

In the following example, we illustrate how the previously presented model can be utilized as a simplified model for a small transportation network:

Example 1: Consider a small part of a transportation network, depicted in Fig. 1. The topology of this transportation network can be modeled by a multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each lane corresponds to a cell and each junction to a node, see Fig. 2. To avoid collisions between vehicles, the local phase matrix can be constructed as follows for node v_1:

$$P^{(v_1)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top ,$$

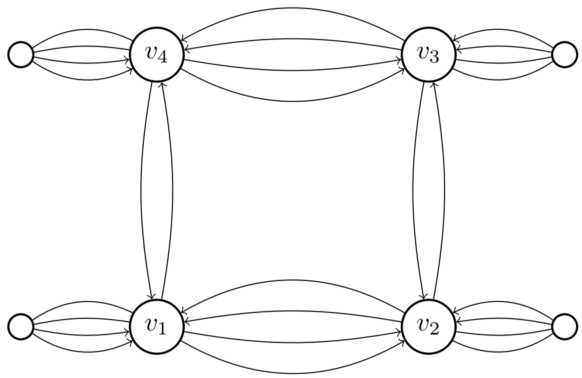

Fig. 2. A graph representation of the transportation network in Fig. 1, consisting of four junctions. Here each node corresponds to one signalized junction. The links correspond to lanes or cells where the vehicles queue up.

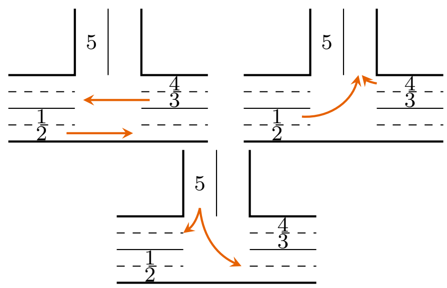

Fig. 3. Example of a local set of phases for junction v_1 in Example 1. In this case, there are three different phases and those phases are orthogonal.

and in similar way for the other nodes. The phases are orthogonal and depicted in Fig. 3.

The next two examples show two different control strategies that fit into our framework.

Example 2 (MaxPressure-control):

$$\omega(x) = \operatorname*{argmax}_{\nu \in \mathcal{U}} \nu^\top P^\top (I - R)x\,, \qquad x \in \mathcal{X}\,. \tag{8}$$

In the above, for each node k in $\mathcal{V}$ and for each local phase p in $\mathcal{P}_k$, we can interpret the quantity

$$s_p^{(k)}(x) = \sum_{i \in \mathcal{E}_k} P_{ip}^{(k)} \left(x_i - \sum_{j \in \mathcal{E}} R_{ij} x_j \right) \tag{9}$$

as the pressure associated with phase p. Then, the MaxPressure controller selects, for each node k in $\mathcal{V}$, the local phase with the maximum pressure. Observe that computing the pressure of a local phase requires measuring the traffic volume on the cells that belong to the local phase itself and on the links immediately downstream, as well as knowledge of the routing matrix.

Example 3 (GPA control with orthogonal phases): For the special case where all the phases are orthogonal, i.e., the phase matrix satisfies $P\mathbb{1} = \mathbb{1}$, we consider the Generalized Proportional Allocation control defined as follows. For every node k in $\mathcal{V}$, fix a $\xi_k > 0$ and, for every local phase p in $\mathcal{P}_k$ and state vector x in $\mathcal{X}$, define

$$\upsilon_p^{(k)}(x) = \frac{\sum_{i \in \mathcal{E}_k} P_{ip}^{(k)} x_i}{\xi_k + \sum_{j \in \mathcal{E}_k} x_j}\,. \tag{10}$$

Then, stack the values $\upsilon_p(x)$ in a vector $\upsilon(x)$ in $\mathcal{U}$ and define the GPA controller as the singleton

$$\omega(x) = \{\upsilon(x)\}\,. \tag{11}$$

Observe that the map $\upsilon : \mathcal{X} \to \mathcal{U}$ defined by (10) is Lipschitz continuous, so that the aforementioned results from [47] can be applied in this case to guarantee existence and uniqueness of solutions of the closed-loop network flow dynamics (5)–(7).

This example also illustrates the need of specifying the flow dynamics (5)–(7) through inequalities. Suppose that two cells i and j belong to the same phase p in $\mathcal{P}$ and $x_i > 0$. Then, if $x_j = 0$, $\zeta_j(x)$ will still be strictly positive, in spite of the fact that cell j is empty. Hence, the outflow z_j has to be such that $z_j < \zeta_j(x)$. In Section IV we shall study a more general form of the GPA controller that applies to arbitrary (i.e., not necessarily orthogonal) phase sets.

III. Fundamental limitations

In this section, we prove a fundamental limitation on the maximal exogenous inflow that a flow network can handle, independently from the control strategy. Specifically, we will introduce a certain *stability region* and prove that it is impossible for any control to stabilize the dynamical flow network when the exogenous inflow is outside such stability region.

We start by introducing the following notion of stability for a dynamical flow network, characterized as the property that for every initial state the traffic volumes remain bounded.

Definition 1 (Stability of a dynamical flow network): Given a flow network $(\mathcal{G}, P)$, an exogenous inflow vector λ, a routing matrix R, an initial state $x(0)$ in $\mathcal{X}$, and control signal $(\mathcal{W}(t))_{t \geq 0}$, a solution of the dynamical flow network (1)–(4) is stable if there exists a positive constant D such that $\|x(t)\| \leq D$ for every $t \geq 0$.
The above definition coincides with the notion of Lagrange stability in dynamical systems theory [48, Ch. III.4] and has been often adopted in the theory of dynamical flow networks [5], [7], [15], [16]. With respect to the stricter notions of Lyapunov stability, it allows for studying a broader set of scenarios including, e.g., time-varying controllers and exogenous inflows inducing trajectories that do not necessarily approach an equilibrium.

We now proceed by introducing the set of feasible flows of a flow network.

Definition 2: The *set of feasible flows* of a flow network with topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, c)$ and phase matrix P is the set

$$\mathcal{Z} = \left\{ z \in \mathbb{R}_+^{\mathcal{E}} \mid 0 \leq z \leq CPu \text{ for some } u \in \mathcal{U} \right\}\,.$$

We will now state a necessary condition for stability of a dynamical flow network that is independent of the chosen control signal. First observe that, for a given constant exogenous inflow λ and routing matrix R such that (λ, R) is in-connected, it is physically intuitive that a necessary condition for stability of the dynamical flow network (1)–(4) with any control is that the pair (λ, R) be out-connected. Indeed, if (λ, R) were not out-connected, there would be constant positive exogenous inflow λ_i in a cell i which cannot flow out of the network. For simplicity of the presentation, we will work with the somewhat stronger assumption that the routing matrix R is out-connected. With this assumption R has spectral radius strictly less than one, see, e.g., [49], which in turn implies that the matrix $I-R$ is invertible with nonnegative inverse

$$(I-R)^{-1} = I + R + R^2 + \dots .$$

Proposition 1 (Necessary condition for stability): Consider a flow network with topology $\mathcal{G}$ and phase matrix P and let $\mathcal{Z}$ be its set of feasible flows. Let R be an out-connected routing matrix and λ be a possibly time-varying exogenous inflow vector. If for an initial state $x(0)$ in $\mathbb{R}_+^{\mathcal{E}}$ and a control signal $(\mathcal{W}(t))_{t\geq 0}$ the dynamical flow network (1)–(4) admits a stable solution, then the average inflow vector $\bar{\lambda}(t) = \frac{1}{t}\int_0^t \lambda(s)\mathrm{d}s$ satisfies

$$\lim_{t\to+\infty} \operatorname{dist}\left((I-R^\top)^{-1}\bar{\lambda}(t), \mathcal{Z}\right) = 0 \,. \qquad (12)$$

In particular, if the exogenous inflow vector λ is constant, then condition (12) simply reads

$$(I-R^\top)^{-1}\lambda \in \mathcal{Z} \,. \qquad (13)$$

Proof: For every $t > 0$ and initial state $x(0)$, we have

$$x(t) = x(0) + t\bar{\lambda}(t) - (I-R^\top)\int_0^t z(s)\mathrm{d}s \,. \qquad (14)$$

Since R is out-connected, its spectral radius is less than one, so the matrix $(I-R^\top)$ is invertible. Multiplying both sides of (14) by $\frac{1}{t}(I-R^\top)^{-1}$ and rearranging terms yields

$$(I-R^\top)^{-1}\bar{\lambda}(t) = \bar{z}(t) + \varepsilon(t) \,, \qquad (15)$$

where

$$\bar{z}(t) = \frac{1}{t}\int_0^t z(s)\mathrm{d}s \,, \qquad \varepsilon(t) = \frac{1}{t}(I-R^\top)^{-1}\left(x(t) - x(0)\right) \,.$$

Since $z(s) \in \mathcal{Z}$ for $0 \leq s \leq t$ and $\mathcal{Z}$ is a convex set, it follows that $\bar{z}(t) \in \mathcal{Z}$. Hence (15) implies that

$$\operatorname{dist}\left((I-R^\top)\bar{\lambda}(t), \mathcal{Z}\right) \leq \|\varepsilon(t)\| \,, \quad t \geq 0 \,. \qquad (16)$$

On the other hand, $x(t)$ is a stable solution of the dynamics (1)–(4), so $x(t)$ remains bounded in $t \geq 0$. This implies that $\|\varepsilon(t)\|$ converges to 0 as t grows large, so that (12) follows from (16). In the special case of constant inflow vector λ, we have $(I-R^\top)^{-1}\bar{\lambda}(t) = \lambda$, so that (12) reduces to (13). ■

The previous result provides a necessary condition for a dynamical flow network to be stable, regardless of the chosen control signal. It means that the flow networks stability region for static inflows is naturally characterized by the set of feasible flows:

Definition 3 (Stability region): Let R be an out-connected routing matrix and λ be a static inflow vector. The pair (λ, R) is said to belong to the stability region, if

$$a = (I-R^\top)^{-1}\lambda \in \operatorname{int}(\mathcal{Z}) \,. \qquad (17)$$

In the case where the inflow vectors λ and the routing matrix R are both constant and such that $(I-R^\top)^{-1}\lambda$ belongs to the set of feasible flows $\mathcal{Z}$, so that there exists some control vector $\overline{u}$ in $\mathcal{U}$ such that $(I-R^\top)^{-1}\lambda < CP\overline{u}$, one could prove that the dynamical flow network with the constant signal control $\mathcal{W}(t) = \{\overline{u}\}$ is stable. However, such static and centralized solution would be highly unfeasible as it would require full knowledge of the exogenous inflows λ and of the routing matrix R (which are seldom constant in time and known in advance), and would lack any robustness. Hence a feedback solution, that requires as little information about the network as possible, is strongly preferable. In the next section, we shall introduce such a decentralized feedback solution and prove that it is maximally stable, i.e., it is able to stabilize the dynamical flow network whenever $(I-R^\top)^{-1}\lambda$ belongs to the interior of the set of feasible flows $\mathcal{Z}$.

IV. Generalized proportional allocation controls and stability

In this section we will construct a decentralized feedback control policy that is able to stabilize the network whenever the necessary condition in Proposition 1 is satisfied. The considered control policy, which we refer to as *Generalized Proportional Allocation* (GPA) control, determines the set $\omega(x)$ through a convex optimization problem, namely

$$\omega(x) = \operatorname*{argmax}_{\nu\in\mathcal{U}} H(x,\nu) \,, \qquad (18)$$

where

$$H(x,\nu) = \sum_{i\in\mathcal{E}} x_i \log\left(CP\nu\right)_i + \sum_{k\in\mathcal{V}} \xi_k \log\left(1 - \mathbb{1}^\top \nu^{(k)}\right) , \qquad (19)$$

In the equation above, for every node k in $\mathcal{V}$, $\nu^{(k)}$ denotes the projection of the vector ν in $\mathcal{U}$ on the local control space $\mathcal{U}_k$. Moreover, ξ in $\mathbb{R}_+^{\mathcal{V}}$ is a vector of parameters, introduced to capture the fact that in many applications it is seldom possible to switch simultaneously between different phases, without loosing some control action during the phase shift. However, the fraction of time when no cell receives service is decreasing with the traffic volume, something that well captures the fact that in applications such as transportation networks, one usually lets the traffic signal cycles be longer when the demand is higher [42]. The objective (19) will also allocate more service to queues that are longer. As we will see later, the objective function in (19) will in certain cases have simple explicit solutions. Later in this section we will show that, while this objective function captures a set of desired properties, it also allows us to prove stability of the dynamical flow network.

The GPA control strategy has several benefits. First of all, it is fully distributed: the control action at each node can be computed separately and using local feedback only. This can be seen by rewriting the expression in (19) as

$$H(x,\nu) = \sum_{k\in\mathcal{V}}\left(\sum_{i\in\mathcal{E}_v} x_i \log\left(C^{(k)}P^{(k)}\nu^{(k)}\right)_i + \xi_k \log\left(1 - \mathbb{1}^\top\nu^{(k)}\right)\right) \qquad (20)$$

where, for every node k in $\mathcal{V}$, $C^{(k)}$ the projection of C on the set of cells $\mathcal{E}_k$. By plugging (20) into (18) one finds that the maximization in the righthand side of the latter can be decoupled into m independent maximizations each over the local control space associated to a node k in $\mathcal{V}$:

$$\omega^{(k)}(x) = \\ \underset{\nu \in \mathcal{U}_k}{\operatorname{argmax}} \sum_{i \in \mathcal{E}_v} x_i \log \left(C^{(k)} P^{(k)} \nu^{(k)}\right)_i + \xi_k \log \left(1 - \mathbb{1}^\top \nu^{(k)}\right).$$

From the above, it is also apparent how the local control $\omega^{(k)}(x)$ depends only on the entries $\{x_i\}_{i \in \mathcal{E}_k}$ of the state vector x that correspond to incoming cells to node k. Moreover, since the optimization problem is convex, it can be solved efficiently for each node k in $\mathcal{V}$.

Notice that, to compute the phase activation, the controller does not require any information about the network topology $\mathcal{G}$, the routing matrix R or the exogenous inflow λ. These facts make the controller robust to perturbations, but it also makes it easier to deploy new controllers into the network, since one does not have to retune the already deployed ones.

While obtaining an explicit solution to the problem (18) may not be possible for general sets of phases, in the relevant special case of orthogonal phases, one gets an explicit solution which turns out to coincide with the one anticipated in Example 3 as stated in the following result:

Lemma 1: If the phases are orthogonal, the GPA controller $\omega(x)$ in (18) is a singleton as given by (10)–(11).
The proof of the lemma is given in Appendix A.

In particular, it follows from Lemma 1 and the considerations done in Example 3 that in the case of orthogonal phases, a solution of the dynamical flow network (5)–(7) with GPA control exists and is unique.

For the general case of non-orthogonal phases, the optimization problem (18) defining the GPA controller remains a convex program, so that in particular $\omega(x)$ is a nonempty compact convex subset of the control set $\mathcal{U}$ for every state vector $x \in \mathcal{X}$. In fact, for all state vectors x all of whose entries x_i are strictly positive the objective function $H(x, \nu)$ in (18) is strictly concave so that $\omega(x) = \{\nu(x)\}$ is a singleton. Moreover, it can be shown that the map $x \mapsto \nu(x)$ is continuous on the positive orthant $\{x \in \mathcal{X} : x_i > 0, \forall i \in \mathcal{E}\}$.

Remark 2: The continuity of the controller cannot be extended to the boundary of the orthant and in fact, it is not always the case that the GPA controller $\omega(x)$ remains a singleton when some entries of the state vector x are equal to 0. This prevents us from applying the existence and uniqueness results in [47], although we conjecture that a solution of the dynamical flow network (5)–(7) with GPA control (18) still exists and is unique even for non-orthogonal phases. We emphasize once more that the main result of the paper, Theorem 1 applies to any solution of the dynamical flow network (5)–(7) with GPA control (18), provided such solution exists and regardless of its uniqueness. In particular, if $x_i = 0$ for a subset of cells, the objective function $H(x, \nu)$ in (18) is not necessary strictly concave anymore, and the set $\omega(x)$ may consist of more than one element, as we will illustrate later in Example 5.

The next theorem states that the GPA controller is able to stabilize the dynamical flow network:

Theorem 1: Consider a flow network with topology $\mathcal{G}$ and phase matrix P and let $\mathcal{Z}$ be its region of feasible flows. Then, for every constant exogenous inflow vector λ and routing matrix R such that (λ, R) is both in- and out-connected and is in the set of feasible flows, every solution $x(t)$ of the dynamical flow network (5)–(7) with GPA control (18) is stable and satisfies

$$\lim_{t \to +\infty} \operatorname{dist}(x(t), \mathcal{X}^*),$$

where

$$\mathcal{X}^* = \left\{x \in \mathcal{X} \mid \zeta(x) \geq a\,,\ x^\top(\zeta(x) - a) \geq 0\right\}. \quad (21)$$

The proof of Theorem 1 is postponed to Section IV-B.

In the case when the phases are orthogonal, it is possible to claim how large the aggregate traffic volumes in each phase will be:

Corollary 1: Consider a flow network with topology $\mathcal{G}$ and phase matrix P such that the phases are orthogonal. Let

$$\rho_p = \max_{i \in p} \frac{a_i}{C_i}\,, \quad \forall p \in \mathcal{P}_k\,,\ k \in \mathcal{V}. \quad (22)$$

be the critical utilization for each phase. Then, for every constant exogenous inflow vector λ and routing matrix R such that (λ, R) is both in- and out-connected and is in the set of feasible flows, every solution $x(t)$ of the dynamical flow network (5)–(7) with GPA control (18) is such that the aggregate traffic volume in each phase satisfies

$$\lim_{t \to +\infty} \sum_{i \in p} x_i(t) = \xi_k \frac{\rho_p}{1 - \sum_q \rho_q}\,, \qquad \forall p \in \mathcal{P}_k,\ k \in \mathcal{V}.$$

Proof: From Theorem 1 it follows that $x(t)$ approaches the set $\mathcal{X}^*$ as t grows large. Let x^* be any point in $\mathcal{X}^*$. First, notice that $\zeta_i(x^*) \geq a_i$ for every cell i in $\mathcal{E}$. Moreover, since the phases are assumed to be orthogonal, for each node k in $\mathcal{V}$ and each phase p in $\mathcal{P}$, it must then hold that $\upsilon_p^{(k)}(x^*) \geq \max_{i \in p} \frac{a_i}{C_i}$.

On the other hand, it follows from the definition of $\mathcal{X}^*$ that if $\upsilon_p^{(k)}(x^*) > \max_{i \in p} \frac{a_i}{C_i}$, then one would have $\sum_{i \in p} x_i^* = 0$, which is a contradiction since $\upsilon_p^{(k)}(0) = 0$. Hence, it must hold that $\upsilon_p^{(k)}(x^*) = \max_{i \in p} \frac{a_i}{C_i}$. By utilizing the explicit expression (10), this can equivalently be written as

$$\frac{\sum_{i \in p} x_i^*}{\sum_{j \in \mathcal{E}_v} x_j^* + \xi_k} = \rho_p\,.$$

Let $\bar{x}_p = \sum_{i \in p} x_i^*$ for all p in $\mathcal{P}_k$. Then the equation above can be rewritten as

$$\bar{x}_p = \rho_p \left(\sum_{q \in \mathcal{P}_k} \bar{x}_q + \xi_k \right),$$

which is in turn equivalent to (22). ∎

Remark 3: Corollary 1 describes the asymptotic behavior of the total traffic volume in all cells for every phase. Hence, it also implies an upper bound on the traffic volume at equilibrium for each cell. Remarkably, the limit of the traffic

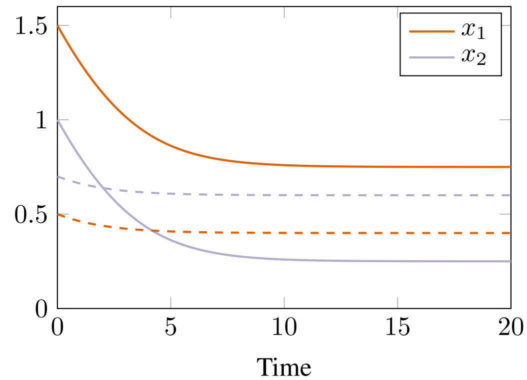

Fig. 4. The state trajectories in Example 4 are plotted in the case $\lambda = 0.5$ and $\xi = 1$, for the initial states $(x_1(0), x_2(0)) = (1.5, 1)$ (solid lines) and $(x_1(0), x_2(0)) = (0.5, 0.7)$ (dashed lines).

volume in all cells for every phase and the resulting upper bound on the traffic volume at equilibrium is only dependent on the exogenous inflows, the routing matrix, the phase matrix, and the flow capacities and holds true irrespective of the initial state $x(0)$.

In the special case when every phase only consists of one cell, i.e. $P^\top \mathbb{1} = \mathbb{1}$, the following corollary states that $\mathcal{X}^*$ is a singleton, something already observed in a more specific setting in [50].

Corollary 2: Consider a flow network with topology $\mathcal{G}$ and phase matrix P such that $P^\top \mathbb{1} = \mathbb{1}$. Then, for every constant exogenous inflow vector λ and routing matrix R such that (λ, R) is both out-connected and in-connected and $a \in \operatorname{int}(\mathcal{Z})$, every solution $x(t)$ of the dynamical flow network (5)–(7) with GPA control (18) converges to a unique point x^* in $\mathcal{X}$, such that $x_i^* > 0$ and $\zeta_i(x) = a_i$ for all i in $\mathcal{E}$.

Proof: This is a consequence of Corollary 1. For every node k in $\mathcal{V}$ and phase p in $\mathcal{P}_k$, the sum $\sum_{i \in p} x_i(t)$ consists of a single addend. Hence $x_i^* > 0$ for all x^* in $\mathcal{X}^*$ and i in $\mathcal{E}$, which combined with the fact that $x^\top(\zeta(x) - a) \geq 0$ implies the result. ∎

A. Examples

The following example shows that for a given set of orthogonal phases, there can be more than one equilibrium point. This justifies why only convergence to the set $\mathcal{X}^*$ can be ensured in Theorem 1.

Example 4: Consider a network with one node and two cells, $\mathcal{E} = \{1, 2\}$, both directed towards the node. The network only has one phase, which both cells belong to. Both of the cells have exogenous inflow $\lambda_1 = \lambda_2 = \lambda > 0$ and $c_1 = c_2 = 1$. Then, the dynamics is given by

$$\dot{x}_1 = \lambda - z_1$$

$$\dot{x}_2 = \lambda - z_2$$

where

$$0 \leq z_1 \leq \upsilon_1(x)\,, \quad x_1(z_1 - \upsilon_1(x)) = 0\,,$$

$$0 \leq z_2 \leq \upsilon_1(x)\,, \quad x_2(z_2 - \upsilon_1(x)) = 0\,,$$

$$\upsilon_1(x) = \frac{x_1 + x_2}{x_1 + x_2 + \xi}\,.$$

If $x_1(0) > x_2(0)$, then $\lim_{t\to+\infty} x_1(t) > \lim_{t\to+\infty} x_2(t)$. On the other hand, if $x_1(0) < x_2(0)$, $\lim_{t\to+\infty} x_1(t) < \lim_{t\to+\infty} x_2(t)$. The trajectories for the two different cases are shown in Fig 4.

The next example illustrates that when the phases are non-orthogonal, the control signal may be set valued:

Example 5: Consider a node k in $\mathcal{V}$ with three cells (indexed $\{1, 2, 3\}$) heading into the node, all with unit capacity. Let the phase matrix be

$$P^{(k)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The maximization problem in (18) can then be equivalently written as

$$\upsilon^{(k)}(x) \in \underset{\nu \in \mathcal{U}_k}{\operatorname{argmax}} \quad x_1 \log(\nu_1) + x_2 \log(\nu_1 + \nu_2) + x_3 \log(\nu_2) + \xi_k \log(1 - \nu_1 - \nu_2)\,.$$

The solution to the maximization problem is:

- If $x_1 = 0, x_2 > 0, x_3 = 0$, then
$$0 \leq \upsilon_1 \leq \frac{x_2}{x_2 + \xi_k}\,, \quad \upsilon_2 = \frac{x_2}{x_2 + \xi_k} - \upsilon_1\,.$$
- For all other cases,
$$\upsilon_1 = \frac{x_1(x_1 + x_2 + x_3)}{(x_1 + x_3)(x_1 + x_2 + x_3 + \xi_k)}\,, \quad \upsilon_2 = \frac{x_3}{x_1}\upsilon_1\,.$$

Let us specifically study the case when $x_1 = x_3 = 0$. In this case, the set $\omega^{(k)}(x)$ is not a singleton anymore. Assume that the cells have exogenous inflows, λ_1, λ_2 and λ_3, respectively, and no inflows from other upstream cells. In this case

$$\upsilon_1 + \upsilon_2 = \frac{x_2}{x_2 + \xi_k}\,.$$

If choosing $\upsilon_1 < \lambda_1$ or $\upsilon_3 < \lambda_3$, then $\dot{x}_1 > 0$ or $\dot{x}_3 > 0$, and the traffic volumes will immediately become positive. Let us for simplicity assume that $\upsilon_1 = 0$ and $\upsilon_3 \geq \lambda_3$, then $\dot{x}_1 > 0$ and after an infinitesimal small time $x_1 > 0$. When this happens, the control signal will be

$$\upsilon_1 = \frac{x_1 + x_2}{x_1 + x_2 + \xi_k} > \lambda_1\,,$$

and x_1 will immediately go back to zero again if $x_2 > \frac{\lambda_1 \xi_k}{1 - \lambda_1}$ is large enough. Therefore trajectory $x(t)$ can not be absolutely continuous in this case. To get an absolutely continuous trajectory $x(t)$ it must hold that $\upsilon_1 > \lambda_1$ and $\upsilon_3 > \lambda_3$ when $x_2 > \frac{\lambda' \xi_k}{1 - \lambda'}$ where $\lambda' = \max(\lambda_1, \lambda_3)$. Recall that $\upsilon_1 > \lambda_1$ and $\upsilon_3 > \lambda_3$ will cause the actual outflow $z_1 < \upsilon_1$ and $z_3 < \upsilon_3$.

We conclude this section by showing how the GPA controller recovers a well-known formula for computing the optimal cycle length in a signalized road traffic junction:

Example 6: Consider a dynamical flow network consisting of one node with two incoming cells $\mathcal{E} = \{1, 2\}$. The exogenous inflows to the cells are $\lambda_1 > 0$, $\lambda_2 > 0$ and their capacities are $c_1 > 0$ and $c_2 > 0$. The node is equipped with

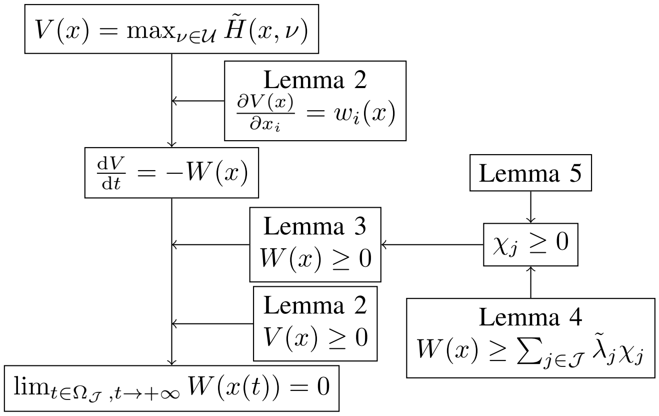

Fig. 5. Scheme of the proof of Theorem 1 illustrating how the different intermediate results contribute to it.

two phases, one for each lane. The dynamics is then described by

$$\dot{x}_1 = \lambda_1 - c_1 \frac{x_1}{x_1+x_2+\xi_k}\,,$$
$$\dot{x}_2 = \lambda_2 - c_2 \frac{x_2}{x_1+x_2+\xi_k}\,.$$

The traffic volumes at equilibrium are

$$(x_1^*, x_2^*) = \left(\frac{\xi_k\rho_1}{1-\rho_1-\rho_2}, \frac{\xi_k\rho_2}{1-\rho_1-\rho_2}\right)\,,$$

where $\rho_i = \lambda_i/c_i$. Observe that the necessary condition for stability is $\rho_1+\rho_2 < 1$. The fraction of the cycle that will be allocated to phase shifts at the equilibrium is then given by

$$\frac{\xi_k}{x_1^*+x_2^*+\xi_k} = \frac{1}{1+\frac{\rho_1}{1-\rho_1-\rho_2}+\frac{\rho_2}{1-\rho_1-\rho_2}} = 1-\rho_1-\rho_2\,.$$

Since the total cycle length will be inverse proportional to the fraction allocated to phase shifts, we get that the cycle length at equilibrium $T(x^*)$ will be proportional to

$$T(x^*) \propto \frac{1}{1-\rho_1-\rho_2}\,.$$

One classical formula for computing the cycle length in a static traffic signal control setting is Webster's formula [44], which suggests that that the cycle length should be

$$T(x^*) = \frac{1.5L+5}{1-\frac{\lambda_1}{c_1}-\frac{\lambda_2}{c_2}}\,,$$

where $L > 0$ is the total loss time, i.e., the total time where no phase is activated. Hence, for any $\xi_k > 0$, the GPA will adjust the cycle length after the demand –without knowing the demand or the lanes outflow capacity– in the same way as Webster's formula suggests.

B. Proof of Theorem 1

The proof of Theorem 1 is provided through a series of intermediate results. How such intermediate results are related to each other and contribute to the proof of the theorem is shown in Fig. 5.

Overall, the proof of Theorem 1 relies on a LaSalle-Lyapunov argument. For every node k in $\mathcal{V}$, let

$$b_k = 1 - \min_{\substack{\nu\in\mathcal{U}_k\,:\\ C^{(k)}P^{(k)}\nu \geq a^{(k)}}} \mathbb{1}^\top\nu\,, \tag{23}$$

and observe that the assumption $a \in \mathrm{int}(\mathcal{Z})$ implies that $b_k > 0$. Then, define the scalar fields

$$\tilde{H} : \mathbb{R}_+^{\mathcal{E}} \times \mathcal{U} \to \mathbb{R}\,, \qquad V : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}\,,$$

by

$$\tilde{H}(x,\nu) = \sum_{i\in\mathcal{E}} x_i \log\frac{(CP\nu)_i}{a_i} + \sum_{k\in\mathcal{V}} \xi_k \log\frac{1-\mathbb{1}^\top\nu^{(k)}}{b_k} \tag{24}$$

and, respectively,

$$V(x) = \max_{\nu\in\mathcal{U}} \tilde{H}(x,\nu)\,. \tag{25}$$

As we shall see, the proof of Theorem 1 relies on showing that, when the generalized proportional allocation feedback controller (10) is employed, the quantity $V(x(t))$ is non-increasing in t along solutions of the network flow dynamics (5)–(7) and strictly decreasing outside the set $\mathcal{X}^*$ defined in (21). Let also $w : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}^{\mathcal{E}}$ be the vector field defined by

$$w_i(x) := \log\left(\frac{\zeta_i(x)}{a_i}\right)\,, \qquad \forall i \in \mathcal{E}\,. \tag{26}$$

The following result gathers a few properties of the functions above.

Lemma 2: Let $\omega(x)$ be the GPA controller defined in (18), and let $H(x,\nu)$, $V(x)$, and $w(x)$ be defined as in (24), (25), and (26), respectively. Then, for every state vector x in $\mathcal{X}$ and control υ in $\omega(x)$,

$$V(x) = \tilde{H}(x,\upsilon) \geq 0\,. \tag{27}$$

Moreover, $V(x)$ is absolutely continuous on $\mathcal{X}$ and

$$\frac{\partial V(x)}{\partial x_i} = w_i(x)\,, \tag{28}$$

for all i such that $x_i > 0$.
Lemma 2 is proved in Appendix A.

A key difficulty in proving that $V(x(t))$ is nondecreasing along solutions $x(t)$ of the network flow dynamics (5)–(7) consists in dealing with the time instants when some of the entries $x_i(t)$ are equal to 0. To address this issue, it proves convenient to introduce the following additional notation. For a state vector x in $\mathcal{X}$, define $\mathcal{I}(x) = \mathcal{I}$ and $\mathcal{J}(x) = \mathcal{J}$ as

$$\mathcal{I} = \{i \in \mathcal{E} \mid x_i = 0\}\,, \qquad \mathcal{J} = \{j \in \mathcal{E} \mid x_j > 0\}\,, \tag{29}$$

and the vector $\tilde{\lambda}(x) \in \mathbb{R}_+^{\mathcal{J}}$, the matrix $\tilde{R}(x) \in \mathbb{R}_+^{\mathcal{J}\times\mathcal{J}}$, and the scalar $W(x)$ in $\mathbb{R}$ as

$$\tilde{\lambda}(x) := \lambda_{\mathcal{J}} + (R^\top)_{\mathcal{J}\mathcal{I}}(I - R_{\mathcal{I}\mathcal{I}}^\top)^{-1}\lambda_{\mathcal{I}}\,, \tag{30}$$

$$\tilde{R}^\top(x) := R_{\mathcal{J}\mathcal{J}}^\top + (R^\top)_{\mathcal{J}\mathcal{I}}(I - R_{\mathcal{I}\mathcal{I}}^\top)^{-1}(R^\top)_{\mathcal{I}\mathcal{J}}\,, \tag{31}$$

and

$$W(x) := -w_{\mathcal{J}}^\top(x)\left(\tilde{\lambda} - (I - \tilde{R}^\top(x))\zeta_{\mathcal{J}}(x)\right)\,, \tag{32}$$

respectively. The following result states a fundamental property of $W(x)$.

Lemma 3: For every state vector x in $\mathcal{X}$, it holds true that

$$W(x) \geq 0$$

with equality if and only if

$$\zeta_{\mathcal{J}}(x) = a_{\mathcal{J}}\,.$$

The proof of Lemma 3 is given in Appendix A.

Proof of Theorem 1: For a state vector x in $\mathcal{X}$, let the subsets of cells $\mathcal{I}(x) = \mathcal{I}$ and $\mathcal{J}(x) = \mathcal{J}$ be defined as in (29). Let $(x(t), z(t))$ be a solution of the dynamics (6)–(7). Observe that, within any open time interval (t_-, t_+) where no entry of $x(t)$ changes sign, so that the sets $\mathcal{I} = \mathcal{I}(x(t))$ and $\mathcal{J} = \mathcal{J}(x(t))$ remain constant, one has that $z_{\mathcal{J}} = \zeta_{\mathcal{J}}(x)$ and

$$0 = \dot{x}_{\mathcal{I}} = \lambda_{\mathcal{I}} + (R^\top)_{\mathcal{I}\mathcal{J}} z_{\mathcal{J}} + R^\top_{\mathcal{I}\mathcal{I}} z_{\mathcal{I}} - z_{\mathcal{I}}$$

so that the vector $z_{\mathcal{I}}$ of outflows from the cells in $\mathcal{I}$ satisfies

$$z_{\mathcal{I}} = (I - R^\top_{\mathcal{I}\mathcal{I}})^{-1}(\lambda_{\mathcal{I}} + (R^\top)_{\mathcal{I}\mathcal{J}}\zeta_{\mathcal{J}}(x)) \tag{33}$$

and the vector $x_{\mathcal{J}}$ of the states of the cells in $\mathcal{J}$ has time-derivative

$$\begin{aligned}\dot{x}_{\mathcal{J}} &= \lambda_{\mathcal{J}} + R^\top_{\mathcal{J}\mathcal{J}}\zeta_{\mathcal{J}}(x) + (R^\top)_{\mathcal{J}\mathcal{I}} z_{\mathcal{I}} \\ &= \tilde{\lambda}(x) - (I - \tilde{R}^\top(x))\zeta_{\mathcal{J}}(x)\,.\end{aligned} \tag{34}$$

Now, let $w : \mathcal{X} \to \mathbb{R}^{\mathcal{E}}$ be the vector field defined by (26) and $V, W : \mathcal{X} \to \mathbb{R}$ be the scalar fields defined by (25) and (32), respectively. Then, for every solution $(x(t), z(t))$ of the dynamics (6)–(7) and for every time instant t belonging to an open interval where the sign of all entries of $x(t)$ are constant, Lemma 2 and (34) imply that

$$\begin{aligned}\dot{V}(x(t)) &= \sum_{j\in\mathcal{J}} \frac{\partial V}{\partial x_j}(x(t))\dot{x}_j(t) \\ &= w^\top_{\mathcal{J}}(x(t))\big(\tilde{\lambda}(x(t)) - (I - \tilde{R}^\top(x(t)))\zeta_{\mathcal{J}}(x(t))\big) \\ &= -W(x(t))\,.\end{aligned}$$

Since $V(x(t))$ is absolutely continuous as a function of t, it follows that

$$V(x(t)) = V(x(0)) - \int_0^t W(x(s))\mathrm{d}s\,.$$

By rearranging terms in the identity above and using Lemma 2 one gets that

$$\int_0^t W(x(s))\mathrm{d}s = V(x(0)) - V(x(t)) \leq V(x(0))\,, \tag{35}$$

for all $t \geq 0$.

Now, it follows from Lemma 3 that

$$W(x(t)) \geq 0\,, \qquad t \geq 0\,. \tag{36}$$

Hence $V(x(t)) \leq V(x(0))$ for all $t \geq 0$.

We will now show that $x(t)$ will be bounded for all $t \geq 0$. Due to the assumption in (17), there exists a $\tilde{\nu}$ in $\mathcal{U}$ such that $(CP\tilde{\nu})_i = a_i(1 + \epsilon_i)$ for some $\epsilon_i > 0$.

$$\begin{aligned}V(x(0)) &\geq V(x(t)) = \max_{\nu\in\mathcal{U}} \tilde{H}(x,\nu) \geq \tilde{H}(x,\tilde{\nu}) \\ &= \sum_{i\in\mathcal{E}} x_i \log(1+\epsilon_i) + D = \sum_{i\in\mathcal{E}} |x_i|\log(1+\epsilon_i) + D\,,\end{aligned}$$

where

$$D = \sum_{k\in\mathcal{V}} \xi_k \log\frac{1 - \mathbb{1}^\top\tilde{\nu}^{(k)}}{b_k}\,.$$

Hence $x(t)$ will be bounded for all $t \geq 0$.

For all $\mathcal{J} \subseteq \mathcal{E}$, let

$$\Omega_{\mathcal{J}} = \mathrm{int}\{t \geq 0 \mid \mathcal{J}(x(t)) = \mathcal{J}\}\,.$$

Now, inequality (36), combined with (35), implies that the integral

$$\int_{\Omega_{\mathcal{J}}} W(x(s))\mathrm{d}s \leq \lim_{t\to+\infty}\int_0^t W(x(s))\mathrm{d}s \leq V(x(0))$$

is finite for all $\mathcal{J} \subseteq \mathcal{E}$.

Since $x(t)$ is bounded and $W(x)$ is continuous, $W(x(t))$ is uniformly continuous on $\Omega_{\mathcal{J}}$. This implies that

$$\lim_{\substack{t\in\Omega_{\mathcal{J}}\\ t\to+\infty}} W(x(t)) = 0\,, \tag{37}$$

for all $\mathcal{J} \subseteq \mathcal{E}$ such that $\Omega_{\mathcal{J}}$ has infinite measure. Then, it follows from (37) and Lemma 3 that

$$\lim_{\substack{t\in\Omega_{\mathcal{J}}\\ t\to+\infty}} \zeta_{\mathcal{J}}(x(t)) = a_{\mathcal{J}}\,. \tag{38}$$

On the other hand, one has that

$$\lambda_{\mathcal{I}} = ((I - R^\top)a)_{\mathcal{I}} = (I - (R^\top)_{\mathcal{I}\mathcal{I}})a_{\mathcal{I}} - (R^\top)_{\mathcal{I}\mathcal{J}}a_{\mathcal{J}}\,. \tag{39}$$

Using (33), (38), and (39), one gets that

$$\begin{array}{rcl}\zeta_{\mathcal{I}}(x(t)) & \geq & z_{\mathcal{I}}(t) \\ & = & (I - R^\top_{\mathcal{I}\mathcal{I}})^{-1}(\lambda_{\mathcal{I}} + (R^\top)_{\mathcal{I}\mathcal{J}}\zeta_{\mathcal{J}}(x)) \\ & \underset{t\in\Omega_{\mathcal{J}}}{\overset{t\to\infty}{\longrightarrow}} & (I - R^\top_{\mathcal{I}\mathcal{I}})^{-1}(\lambda_{\mathcal{I}} + (R^\top)_{\mathcal{I}\mathcal{J}}a_{\mathcal{J}}) \\ & = & a_{\mathcal{I}}\,.\end{array} \tag{40}$$

Together, (38) and (40) imply that

$$\liminf_{\substack{t\in\Omega_{\mathcal{J}}\\ t\to+\infty}} \zeta(x(t)) \geq a\,,$$

so that, for every $\mathcal{J} \subseteq \mathcal{E}$ such that $\Omega_{\mathcal{J}}$ has infinite measure,

$$\lim_{\substack{t\in\Omega_{\mathcal{J}}\\ t\to+\infty}} \mathrm{dist}\,(x(t), \mathcal{X}^*) = 0\,.$$

The claim now follows from the fact that, on the one hand, since $x(t)$ is absolutely continuous,

$$\mathbb{R}_+ = \bigcup_{\mathcal{J}\subset\mathcal{E}} \Omega_{\mathcal{J}} \cup A$$

for some measure-0 subset of times $A \subseteq \mathbb{R}_+$, on the other hand,

$$\lim_{t\to+\infty} \mu(\Omega_{\mathcal{J}} \cap [t, +\infty)) = 0$$

for every $\mathcal{J} \subseteq \mathcal{E}$ such that $\Omega_{\mathcal{J}}$ has finite measure. ■

TABLE I
THE NON-ZERO ENTRIES IN THE ROUTING MATRIX

Original values			
v_1	v_2	v_3	v_4
$R_{1,20} = 1$	$R_{6,15} = 1$	$R_{11,10} = 1$	$R_{16,5} = 1$
$R_{2,6} = 0.2$	$R_{8,3} = 0.7$	$R_{12,16} = 0.4$	$R_{18,13} = 0.5$
$R_{2,7} = 0.8$	$R_{8,4} = 0.3$	$R_{12,17} = 0.6$	$R_{18,14} = 0.5$
$R_{4,20} = 1$	$R_{9,15} = 1$	$R_{14,10} = 1$	$R_{19,5} = 1$
$R_{5,6} = 0.07$	$R_{10,3} = 0.245$	$R_{15,16} = 0.14$	$R_{20,13} = 0.175$
$R_{5,7} = 0.28$	$R_{10,20} = 0.105$	$R_{15,17} = 0.21$	$R_{20,14} = 0.175$

New values (for cell $5, 10, 15, 20$)			
v_1	v_2	v_3	v_4
$R_{5,6} = 0.1$	$R_{10,3} = 0.35$	$R_{15,16} = 0.2$	$R_{20,13} = 0.25$
$R_{5,7} = 0.4$	$R_{10,20} = 0.15$	$R_{15,17} = 0.3$	$R_{20,14} = 0.25$

V. Numerical Simulation

To illustrate the concepts presented in this paper, we will simulate the dynamical system with the topology shown in Fig. 2. For each of the four nodes, we let the set of phases be the same as in Example 1. We let the exogenous inflow rate be 0.2 on all incoming cells from the outside of the network, i.e., the cells 1, 2, 8, 9, 11, 12, 18, and 19. For simplicity of the presentation, we assume identical outflow capacity from all cells, and normalize all the units in such a way that the outflow capacity is 1 for every cell in the network. The non-zero entries of the routing matrix are shown in Table I.

To illustrate the controller's ability to adapt a new routing setting and hence the controller's robustness to demand changes within the set of feasible flows, we will change the routing matrix during the simulation. When one-third of the simulation time has passed we change some of the entries in the routing matrix, related to the outflows as specified in the lower part of Table I.

The trajectories for the dynamics (6)–(7) with GPA control (10) in the setting previously described are shown in Fig. 6. For all four nodes, we let the initial traffic volume on the incoming cells be $x^{(v_1)}(0) = x^{(v_2)}(0) = x^{(v_3)}(0) = x^{(v_4)}(0) = (0.5, 0.4, 0.3, 0.2, 0.1)$. As we can see, the controller manages to keep the traffic volumes bounded, and adapt to a new setting when the routing is changed. We also see that a few cells will stay around zero traffic volume. This is expected, since we have cells with different average inflow rates belonging to the same phase, so the cell with a lower average inflow rate will stay at zero.

In Fig. 7 we show the control signals, together with the average inflow rates, we see that the control signals are always greater than or equal to the average inflow rates, something that is necessary to keep the traffic volumes bounded. For the cells where the control signals are strictly greater than the average inflow rates, the traffic volume will stay zero and the actual outflow from every such cell will equal its inflow. We also see that the controller is able to adapt the new setting when the routing has changed, without having any information about that the routing has changed. This illustrates the robustness of the proposed controller.

For reference, we also simulate a static controller. For the static controller, we have chosen the control signals to be to be $\nu^{(1)} = (0.21, 0.25, 0.31)$, $\nu^{(2)} = (0.21, 0.28, 0.37)$, $\nu^{(3)} = (0.21, 0.21, 0.27)$ and $\nu^{(4)} = (0.21, 0.21, 0.31)$. While these

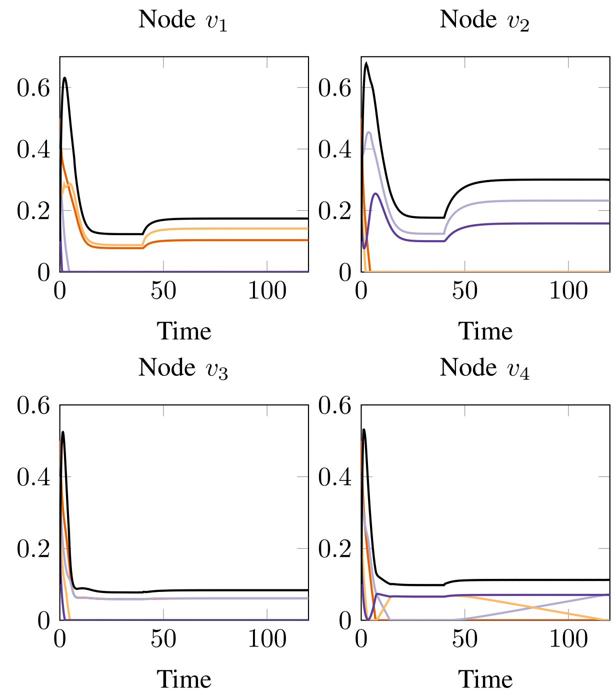

Fig. 6. The traffic volumes when the network in Section V is controlled with the GPA controller. The coloring is the following: Cell $1, 6, 11, 16$ - (——), Cell $2, 7, 12, 17$ - (——), Cell $3, 8, 13, 18$ - (——), Cell $4, 9, 14, 19$ - (——), and Cell $5, 10, 15, 20$ - (——).

control signals are appropriate to handle the traffic flows before the perturbation, they cause instability after the perturbation. How the trajectories evolve with time for this example, when the control signal is static, is shown in Fig. 8.

VI. Conclusion

We have presented a feedback based service allocation policy for dynamical flow networks that is decentralized as the service allocation in each part of the network only depends on the queue lengths in that part of the network. Moreover, the policy does not require any information on the network topology, the exogenous inflows, or the routing matrix. Despite its information frugality, the controller stabilizes the dynamical flow network, whenever any controller can do so.

Future work includes investigating the stability properties of the GPA with more complex flow dynamics tailored specifically for the relevant application, including propagation delay, congestion effects, and limited storage capacity, as well as considering time-varying routing matrices.

Appendix

A. Proofs of Lemmas

For the reader's convenience, the statements of the lemmas are included in this appendix as well.

Lemma 1: If the phases are orthogonal, the GPA control $\omega(x)$ in (18) is a singleton as given by (10)–(11).

Proof: To show that (10) is a solution to (18), we have to show that

$$\omega(x(t)) = \operatorname*{argmax}_{v \in \mathcal{U}} H(x(t), v) \,. \tag{41}$$

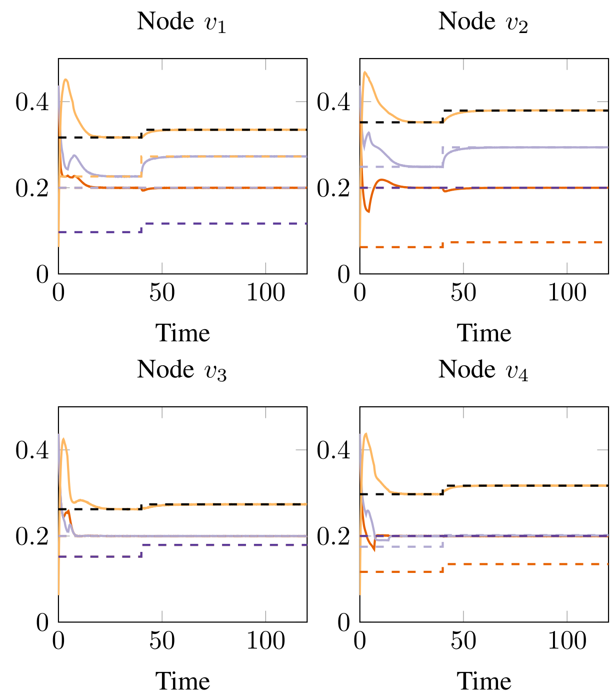

Fig. 7. The control signals when the network in Section V is controlled with the GPA controller. The phase activation for Phase 1 is shown in (——), for Phase 2 in (——), and Phase 3 in (——). The dashed lines are the average arrival rates with the coloring: Cell $1, 6, 11, 16$ - (- - -), Cell $2, 7, 12, 17$ - (- - -), Cell $3, 8, 13, 18$ - (- - -), Cell $4, 9, 14, 19$ - (- - -), and Cell $5, 10, 15, 20$ - (- - -).

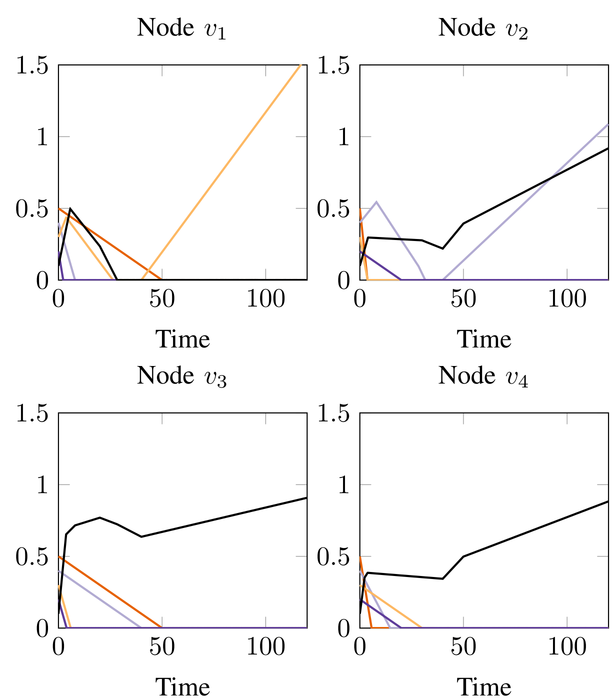

Fig. 8. The traffic volumes when the network in Section V is controlled with the static controller. The coloring is the following: Cell $1, 6, 11, 16$ - (——), Cell $2, 7, 12, 17$ - (——), Cell $3, 8, 13, 18$ - (——), Cell $4, 9, 14, 19$ - (——), and Cell $5, 10, 15, 20$ - (——).

Define the the Lagrangian $L : \mathcal{X} \times \mathcal{U} \times \mathbb{R}_+^{\mathcal{V}} \to \mathbb{R}$ associated with the optimization problem in (41) as

$$\begin{aligned} L(x, \upsilon, \gamma) &= H(x, \upsilon) + \sum_{k \in \mathcal{V}} \gamma_k (1 - \mathbb{1}^\top \upsilon^{(k)}) \\ &= \sum_{i \in \mathcal{E}} x_i \log(CP\upsilon)_i + \sum_{k \in \mathcal{V}} \xi_k \log 1 - \mathbb{1}^\top \upsilon^{(k)} \\ &\quad + \sum_{k \in \mathcal{V}} \gamma_k (1 - \mathbb{1}^\top \upsilon^{(k)}) \\ &= \sum_{k \in \mathcal{V}} \Bigg(\sum_{i \in \mathcal{E}_v} x_i \log(CP\upsilon)_i \\ &\quad + \xi_k \log 1 - \mathbb{1}^\top \upsilon^{(k)} + \gamma_k (1 - \mathbb{1}^\top \upsilon^{(k)}) \Bigg) , \end{aligned}$$

where γ in $\mathbb{R}_+^{\mathcal{V}}$ is the vector of Lagrange multipliers associated to the inequality constraints $\mathbb{1}^\top \upsilon^{(k)} \leq 1$. Then, necessary conditions for optimum are that

$$\frac{\partial L}{\partial \upsilon_q^{(k)}} = \frac{1}{\upsilon_q^{(k)}} \sum_{i \in \mathcal{E}_v} P_{iq}^{(k)} x_i - \frac{1}{1 - \mathbb{1}^\top \upsilon^{(k)}} \xi_k - \gamma_k = 0 ,$$

for all k in $\mathcal{V}$ and q in $\mathcal{P}_k$. Moreover, since the problem in (41) is convex, using the complementary slackness principle [51], we get that either $1 - \mathbb{1}^\top \upsilon^{(k)}$ is zero, which clearly cannot be a maximum, or $\gamma_k = 0$. For the latter case, it holds that

$$\frac{1}{\xi_k} \sum_{i \in \mathcal{E}_k} P_{iq}^{(k)} x_i = \frac{\upsilon_q^{(k)}}{1 - \mathbb{1}^\top \upsilon^{(k)}} . \tag{42}$$

Summing up the expression above over all phases q in $\mathcal{P}_v$ and using the fact that the phases are orthogonal yields

$$\frac{1}{\xi_k} \sum_{i \in \mathcal{E}_k} x_i = \frac{\mathbb{1}^\top \upsilon^{(k)}}{1 - \mathbb{1}^\top \upsilon^{(k)}} ,$$

and hence

$$\mathbb{1}^\top \upsilon^{(k)} = \frac{\sum_{i \in \mathcal{E}_k} x_i}{\xi_k + \sum_{i \in \mathcal{E}_k} x_i} . \tag{43}$$

By combining (42) and (43) we get

$$\upsilon_q^{(k)} = \frac{\sum_{i \in \mathcal{E}_k} P_{iq} x_i}{\xi_k + \sum_{i \in \mathcal{E}_k} x_i} ,$$

which, together with the concavity of (19), proves that (10) is a solution to (18). ■

Lemma 2: Let $\omega(x)$ be the GPA controller defined in (18), and let $H(x, \nu)$, $V(x)$, and $w(x)$ be defined as in (24), (25), and (26), respectively. Then, for every state vector x in $\mathcal{X}$ and control υ in $\omega(x)$,

$$V(x) = \tilde{H}(x, \upsilon) \geq 0 . \tag{27}$$

Moreover, $V(x)$ is absolutely continuous on $\mathcal{X}$ and

$$\frac{\partial V(x)}{\partial x_i} = w_i(x) , \tag{28}$$

for all i such that $x_i > 0$.

Proof: The equality in (27), that

$$\max_{\nu \in \mathcal{U}} \tilde{H}(x, \nu) = \tilde{H}(x, \upsilon)$$

is a solution to (18) follows from the fact that

$$\operatorname*{argmax}_{\nu\in\mathcal{U}} \tilde{H}(x,\nu) = \operatorname*{argmax}_{\nu\in\mathcal{U}} H(x,\nu)\,,$$

where $H(x,\nu)$ is the expression in (19).

The inequality in (27) stating that $V(x) \geq 0$ follows from the fact that

$$V(x) = \max_{\nu\in\mathcal{U}} \tilde{H}(x,\nu) \geq \tilde{H}(x,\tilde{\nu}) \geq 0\,,$$

where $\tilde{\nu} \in \mathcal{U}$ is chosen such that $(CP\tilde{\nu})_i \geq a_i$ for all i in $\mathcal{E}$ and $1 - \mathbb{1}^\top \tilde{\nu}^{(k)} = b_k$ for all k in $\mathcal{V}$. It follows from the definition of b_k in (23) that this choice of $\tilde{\nu}$ is feasible.

To show (28), we follow the idea presented in [41]. For a state vector x in $\mathcal{X}$ and i in $\mathcal{E}$, let $x^{(\epsilon)}$ in $\mathcal{X}$ be a vector such that $x_i^{(\epsilon)} = x_i + \epsilon$ for some $\epsilon > 0$ and $x_j^{(\epsilon)} = x_j$ for all $j \neq i$ in $\mathcal{E}$. Then

$$\begin{aligned}
V(x^{\epsilon}) - V(x) = &\\
&\sum_{j\in\mathcal{E}} x_j^{(\epsilon)} \log \frac{\zeta_j(x^{(\epsilon)})}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x^{(\epsilon)})}{b_k}\\
&\quad - \sum_{j\in\mathcal{E}} x_j \log \frac{\zeta_j(x)}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x)}{b_k}\\
&\geq \sum_{j\in\mathcal{E}} x_j^{(\epsilon)} \log \frac{\zeta_j(x)}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x)}{b_k}\\
&\quad - \sum_{j\in\mathcal{E}} x_j \log \frac{\zeta_j(x)}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x)}{b_k}\\
&\quad = \epsilon \log \frac{\zeta_i(x)}{a_i}\,,
\end{aligned}$$

where the inequality follows from the fact that

$$H(x^{(\epsilon)}, \upsilon(x^{(\epsilon)})) = \max_{\nu\in\mathcal{U}} H(x^{(\epsilon)},\nu) \geq H(x^{(\epsilon)}, \upsilon(x))\,.$$

In the same manner, we have that

$$\begin{aligned}
V(x^{(\epsilon)}) - V(x) = &\\
&\sum_{j\in\mathcal{E}} x_j^{(\epsilon)} \log \frac{\zeta_j(x^{(\epsilon)})}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x^{(\epsilon)})}{b_k}\\
&\quad - \sum_{j\in\mathcal{E}} x_j \log \frac{\zeta_j(x)}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x)}{b_k}\\
&\leq \sum_{j\in\mathcal{E}} x_j^{(\epsilon)} \log \frac{\zeta_j(x^{(\epsilon)})}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x^{(\epsilon)})}{b_k}\\
&\quad - \sum_{j\in\mathcal{E}} x_j \log \frac{\zeta_j(x^{(\epsilon)})}{a_j} + \sum_{k\in\mathcal{V}} \xi_k \log \frac{1-\mathbb{1}^\top \upsilon^{(k)}(x^{(\epsilon)})}{b_k}\\
&\quad = \epsilon \log \frac{\zeta_i(x^{(\epsilon)})}{a_i}\,.
\end{aligned}$$

The bounds combined together yields

$$\log \frac{\zeta_i(x)}{a_i} \leq \frac{1}{\epsilon}(V(x^{(\epsilon)}) - V(x)) \leq \log \frac{\zeta_i(x^{(\epsilon)})}{a_i}\,.$$

Since the optimization problem in (18) is strictly concave for all $x > 0$, it follows from the maximum theorem [52, Theorem 9.14], that $\upsilon(x)$ depends continuously on x. Hence $\zeta(x)$ depends continuously on x, letting $\epsilon \to 0$ proves the last statement of the lemma. ■

Lemma 3: For every state vector x in $\mathcal{X}$, it holds true that

$$W(x) \geq 0$$

with equality if and only if

$$\zeta_{\mathcal{J}}(x) = a_{\mathcal{J}}\,.$$

We prove Lemma 3 by combining two intermediate results. The first one is a lower bound on $W(x)$ as stated in the following.

Lemma 4: For every state x in $\mathcal{X}$ we have

$$W(x) = \sum_{j\in\mathcal{J}} \tilde{\lambda}_j F_j(w_{\mathcal{J}})\,, \tag{44}$$

where

$$F(w_{\mathcal{J}}) = (I-\tilde{R})^{-1} \operatorname{diag}\left((I-\tilde{R})w_{\mathcal{J}}\right)(e^{w_{\mathcal{J}}} - \mathbb{1})\,,$$

and $e^{w_{\mathcal{J}}}$ is the vector with entries $(e^{w_{\mathcal{J}}})_j = e^{w_j}$ for j in $\mathcal{J}$. Moreover,

$$F_j(w_{\mathcal{J}}) \geq \chi_j\,, \qquad \forall j \in \mathcal{J}\,, \tag{45}$$

where

$$\chi_j = \sum_{i,k\in\mathcal{J}} N_{ik}^{(j)} w_i (e^{w_i} - 1) - \sum_{i,k\in\mathcal{J}} N_{ik}^{(j)} w_i (e^{w_k} - 1)$$

and, for every i, j, k in $\mathcal{J}$,

$$N_{ik}^{(j)} = \sum_{h\geq 0} \tilde{R}_{ji}^h \left(\tilde{R}_{ik} + \delta_k^{(j)}\Big(1 - \sum_{l\in\mathcal{J}} \tilde{R}_{il}\Big)\right)\,. \tag{46}$$

Proof: It follows from $\lambda = (I - R^\top)a$ that

$$\begin{aligned}
\lambda_{\mathcal{I}} &= (I - R_{\mathcal{I}\mathcal{I}}^\top)a_{\mathcal{I}} - (R^\top)_{\mathcal{I}\mathcal{J}} a_{\mathcal{J}}\,,\\
\lambda_{\mathcal{J}} &= (I - R_{\mathcal{J}\mathcal{J}}^\top)a_{\mathcal{J}} - (R^\top)_{\mathcal{J}\mathcal{I}} a_{\mathcal{I}}\,.
\end{aligned}$$

Using the above, as well as (30), we obtain that

$$\begin{aligned}
(I - R_{\mathcal{J}\mathcal{J}}^\top)a_{\mathcal{J}} &= \lambda_{\mathcal{J}} + (R^\top)_{\mathcal{J}\mathcal{I}} a_{\mathcal{I}}\\
&= \tilde{\lambda} + (R^\top)_{\mathcal{J}\mathcal{I}}(I - R_{\mathcal{I}\mathcal{I}}^\top)^{-1}(R^\top)_{\mathcal{I}\mathcal{J}} a_{\mathcal{J}}
\end{aligned}$$

so that, by substituting (31), we get that

$$(I - \tilde{R}^\top)a_{\mathcal{J}} = \tilde{\lambda}\,.$$

Let $A = \operatorname{diag}(a_{\mathcal{J}})$. Then, $\zeta_{\mathcal{J}}(x) = Ae^{w_{\mathcal{J}}}$, so that

$$\begin{aligned}
W(x) &= -w_{\mathcal{J}}^\top \left(\tilde{\lambda} - (I - \tilde{R}^\top)Ae^{w_{\mathcal{J}}}\right)\\
&= -w_{\mathcal{J}}^\top \left((I - \tilde{R}^\top)A\mathbb{1} - (I - \tilde{R}^\top)Ae^{w_{\mathcal{J}}}\right)\\
&= -w_{\mathcal{J}}^\top (I - \tilde{R}^\top)A(\mathbb{1} - e^{w_{\mathcal{J}}})\\
&= \tilde{\lambda}^\top F(w_{\mathcal{J}})\,,
\end{aligned}$$

which proves the first part of the claim.

In order to prove the second part, let

$$B(w_{\mathcal{J}}) = \operatorname{diag}\left((I-\tilde{R})w_{\mathcal{J}}\right)(e^{w_{\mathcal{J}}} - \mathbb{1})\,.$$

For i in $\mathcal{J}$, rewrite $w_i = [w_i]_+ - [w_i]_-$ and observe that $e^{[w_i]_\pm} - 1 = [q_i]_\pm$, where $q_i = e^{w_i} - 1$. Then,

$$\begin{aligned} B_i(w_{\mathcal{J}}) &= q_i \left(w_i - \textstyle\sum_k \tilde{R}_{ik} w_k \right) \\ &= [q_i]_+ \big([w_i]_+ - \textstyle\sum_k \tilde{R}_{ik}[w_k]_+ \big) \\ &\quad + [q_i]_- \big([w_i]_- - \textstyle\sum_k \tilde{R}_{ik}[w_k]_+ \big) \\ &\quad + [q_i]_+ \textstyle\sum_k R_{ik}[w_k]_- + [q_i]_- \textstyle\sum_k R_{ik}[w_k]_- \\ &\geq B_i(w^+_{\mathcal{J}}) + B_i(w^-_{\mathcal{J}}) \,, \end{aligned}$$

where the summation index k is intended to run over $\mathcal{J}$ and the fact that $[q_i]_\pm [w_i]_\mp = 0$ is used in the second equality. Since $(I - \tilde{R})^{-1}$ is a nonnegative matrix, the above implies that

$$\begin{aligned} F(w_{\mathcal{J}}) &= (I - \tilde{R})^{-1} B(w_{\mathcal{J}}) \\ &\geq (I - \tilde{R})^{-1} B([w_{\mathcal{J}}]_+) + (I - \tilde{R})^{-1} B([w_{\mathcal{J}}]_-) \\ &= F([w_{\mathcal{J}}]_+) + F([w_{\mathcal{J}}]_-) \,. \end{aligned}$$

Now, rewrite $F_j(w_{\mathcal{J}})$ as

$$\begin{aligned} F_j(w_{\mathcal{J}}) &= \sum_{i \in \mathcal{J}} \sum_{n \geq 0} \tilde{R}^n_{ji} (w_i - \sum_{k \in \mathcal{J}} \tilde{R}_{ik} w_k)(e^{w_i} - 1) \\ &= \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{w_i} - 1) w_i - \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{w_i} - 1) w_k \\ &\quad - \sum_{i \in \mathcal{J}} \sum_{n \geq 0} \tilde{R}^n_{ji} \left(1 - \sum_{l \in \mathcal{J}} \tilde{R}_{il} \right) (e^{w_i} - 1) w_j \,. \end{aligned} \tag{47}$$

It then follows that

$$\begin{aligned} F_j(w_{\mathcal{J}}) &\geq F_j([w_{\mathcal{J}}]_+) + F_j([w_{\mathcal{J}}]_-) \\ &\geq \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{[w_i]_+} - 1)[w_i]_+ \\ &\quad - \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{[w_i]_+} - 1)[w_k]_+ \\ &\quad + \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{[w_i]_-} - 1)[w_i]_- \\ &\quad - \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{[w_i]_-} - 1)[w_k]_- \\ &\geq \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{w_i} - 1) w_i - \sum_{i,k \in \mathcal{J}} N^{(j)}_{ik} (e^{w_i} - 1) w_k \\ &= \chi_j \,, \end{aligned}$$

thus completing the proof. ■

Lemma 5: Let μ in $\mathbb{R}^n_+$ be a vector with strictly positive entries and let

$$\mathcal{M} := \left\{ M \in \mathbb{R}^{n \times n}_+ \mid M\mathbb{1} = M^\top \mathbb{1} = \mu \right\}$$

be the set of nonnegative square matrices with both row and column sum vectors equal to μ. Let $f, g : \mathbb{R} \to \mathbb{R}$ be strictly increasing functions. Then, for every vector v in $\mathbb{R}^n$, it holds true that

$$\sum_{i=1}^n \mu_i f(v_i) g(v_i) \geq \sum_{i=1}^n \sum_{j=1}^n M_{ij} f(v_i) g(v_j) \,, \tag{48}$$

for every M in $\mathcal{M}$, with equality if and only if

$$M_{ij} = 0 \,, \qquad \forall\, i, j : \, v_i \neq v_j \,. \tag{49}$$

Proof: Let us define the function $h : \mathcal{M} \to \mathbb{R}$ by

$$h(M) = \sum_{i=1}^n \sum_{j=1}^n M_{ij} f(v_i) g(v_j) \,.$$

Observe that $h(M)$ is a continuous function and $\mathcal{M}$ is a compact set. Hence, $h(M)$ admits a maximum over $\mathcal{M}$. We shall prove the claim by showing that such maximum value is

$$\max\{ h(M) \mid M \in \mathcal{M} \} = \sum_{i=1}^n \mu_i f(v_i) g(v_i)$$

and that the set of maximum points

$$\operatorname{argmax}\{ h(M) \mid M \in \mathcal{M} \} = \{ M \in \mathcal{M} \mid (49) \}$$

coincides with the subset of matrices satisfying (49).

Without any loss of generality, we shall assume that

$$v_1 \leq v_2 \leq \dots \leq v_{n-1} \leq v_n \,.$$

Now, let $m \leq n$ be the number of distinct entries of v and let $\mathcal{H}_1, \dots, \mathcal{H}_m \subseteq \{1, \dots, n\}$ be the subsets of indices such that $v_i = v_j$ if and only if i, j in $\mathcal{H}_l$ for the same $1 \leq l \leq m$. Then, a matrix M in $\mathcal{M}$ satisfies (49) if and only if is in the following block diagonal form

$$M = \begin{bmatrix} M^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M^{(m)} \end{bmatrix} \,,$$

with each block $M^{(l)}$ in $\mathbb{R}^{|\mathcal{H}_l| \times |\mathcal{H}_l|}_+$ for $1 \leq l \leq m$. Using the block diagonal form above, for an arbitrary selection of k_l in $\mathcal{H}_l$, $1 \leq l \leq m$, one gets that

$$\begin{aligned} h(M) &= \sum_{l=1}^m \sum_{i,j \in \mathcal{H}_l} M_{ij} f(v_i) g(v_j) \\ &= \sum_{l=1}^m |\mathcal{H}_l| \mu_{k_l} f(v_{k_l}) g(v_{k_l}) \\ &= \sum_{i=1}^n \mu_i f(v_i) g(v_i) \,, \end{aligned} \tag{50}$$

for every matrix M in $\mathcal{M}$ satisfying (49).

We are then left with proving that no matrix M in $\mathcal{M}$ not satisfying (49) can be a maximizer of $h(M)$ over $\mathcal{M}$. For any such M, let j be the unique value in $\{1, 2, \dots, n-1\}$ such that $M_{ii} = \mu_i$ for all $1 \leq i < j$ and $M_{jj} < \mu_j$ and let $1 \leq q \leq m$ be such that j in $\mathcal{H}_q$. Then, since M belongs to $\mathcal{M}$ and it does not satisfy (49), there must exist indices k in $\mathcal{H}_r$ and l in $\mathcal{H}_s$, with r, s in $\{q+1, \dots, m\}$, such that

$$\epsilon = \min\{ M_{jl}, M_{kj} \} > 0 \,.$$

Define the matrix $\tilde{M}$ in $\mathbb{R}^{n \times n}$ with entries

$$\tilde{M}_{hi} = \begin{cases} M_{hi} + \epsilon & \text{if } i = j \text{ and } h = j \,, \\ M_{hi} + \epsilon & \text{if } i = l \text{ and } h = k \,, \\ M_{hi} - \epsilon & \text{if } i = l \text{ and } h = j \,, \\ M_{hi} - \epsilon & \text{if } i = j \text{ and } h = k \,, \\ M_{hi} & \text{otherwise} \,. \end{cases}$$

It is easily verified that $\tilde{M}$ in $\mathcal{M}$. Moreover, since j in $\mathcal{H}_q$, k in $\mathcal{H}_r$, and l in $\mathcal{H}_s$, with r, s in $\{q+1, \dots, m\}$, we have that $v_k > v_j$ and $v_l > v_j$. Since the functions f and g are strictly increasing, this implies that

$$f(v_l) > f(v_j)\,, \qquad g(v_k) > g(v_j)\,.$$

It follows that

$$\begin{aligned} 0 &< \epsilon(f(v_l) - f(v_j))(g(v_k) - g(v_j)) \\ &= \epsilon(f(v_j)g(v_j) + f(v_l)g(v_k) - f(v_l)g(v_j) - f(v_j)g(v_k)) \\ &= h(\tilde{M}) - h(M)\,. \end{aligned}$$

The above shows that no matrix M in $\mathcal{M}$ that does not satisfy (49) can be a maximizer of $h(M)$ over $\mathcal{M}$, thus completing the proof. ∎

We are now ready to prove Lemma 3. For i, j, k in $\mathcal{J}$, let $N_{ik}^{(j)}$ be defined as in (46) and let

$$\mu_i^{(j)} = \sum_{h \geq 0} \tilde{R}_{ji}^h\,.$$

Clearly, $\mu_j^{(j)} \geq R_{jj}^0 = 1 > 0$ and, more in general, $\mu_k^{(j)} > 0$ if and only if k is reachable from j through $\tilde{R}$. Let $\mathcal{K}_j$ be the set reachable from j through $\tilde{R}$. Now observe that, for i in $\mathcal{K}_j$,

$$\sum_{k \in \mathcal{K}_j} N_{ik}^{(j)} = \sum_{h \geq 0} \tilde{R}_{ji}^h = \mu_i^{(j)}\,,$$

while, for k in $\mathcal{K}_j$,

$$\sum_{i \in \mathcal{K}_j} N_{ik}^{(j)} = \sum_{h \geq 0} \tilde{R}_{jk}^{h+1} + \sum_{h \geq 0} (\tilde{R}_{jk}^h - \tilde{R}_{jk}^{h+1}) = \mu_k^{(j)}\,.$$

On the other hand, observe that, since $\mathcal{K}_j$ is the set reachable from j, the restriction of the matrix $N^{(j)}$ to $\mathcal{K}_j \times \mathcal{K}_j$ consists of a single diagonal block. Then, (45) and Lemma 5 imply that, for every j in $\mathcal{J}$,

$$\begin{aligned} F_j(w_{\mathcal{J}}) &\geq \chi_j \\ &= \sum_{i,k \in \mathcal{K}_j} N_{ik}^{(j)} w_i (e^{w_i} - 1) - \sum_{i,k \in \mathcal{K}_j} N_{ik}^{(j)} w_i (e^{w_k} - 1) \\ &\geq 0\,, \end{aligned} \tag{51}$$

where the last inequality holds true as an equality if and only if w is constant over $\mathcal{K}_j$. Observe that, in this case, there exists some constant c in $\mathbb{R}$ such that

$$F_j(w_{\mathcal{J}}) = \sum_{i \in \mathcal{K}_j} \sum_{h \geq 0} \tilde{R}_{ji}^h \Big(1 - \sum_{l \in \mathcal{K}_j} \tilde{R}_{il}\Big)(e^c - 1)c \geq 0\,. \tag{52}$$

However, since $\mathcal{K}_j$ is out-connected, then necessarily there must exist at least one i in $\mathcal{K}_j$ such that $\sum_{l \in \mathcal{K}_j} \tilde{R}_{il} < 1$ and an $h \geq 0$ such that $\tilde{R}_{ji}^h > 0$. It then follows from (51) and (52) that

$$F_j(w_{\mathcal{J}}) \geq 0\,, \qquad \forall j \in \mathcal{J}\,, \tag{53}$$

with equality if and only if $w_i = 0$ for every i in $\mathcal{K}_j$.

Finally, observe that

$$\bigcup_{j \in \mathcal{J} : \tilde{\lambda}_j > 0} \mathcal{K}_j = \mathcal{J}\,.$$

The above, (44), and (52) imply that

$$W(x) = \sum_{j \in \mathcal{J}} \tilde{\lambda}_j F_j(w_{\mathcal{J}}) \geq 0\,,$$

with equality if and only if $w_i = 0$ for all i in $\mathcal{J}$, i.e., if and only if

$$\zeta_i(x) = a_i\,, \qquad \forall i \in \mathcal{J}\,.$$

The proof of Lemma 3 is then complete. ∎

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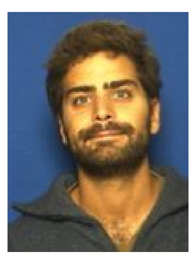

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