On the well-posedness of deterministic queuing networks with feedback control *

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Abstract

We study the well-posedness of a class of dynamical flow network systems describing the dynamical mass balance among a finite number of cells exchanging flow of traffic between themselves and with the external environment. Dynamical systems in the considered class are described as differential inclusions whereby the routing matrix is constant and the outflow from each cell in the network is limited by a control that is a Lipschitz continuous function of the state of the network. This framework finds application in particular within traffic signal control, whereby it is common that an empty queue can be allowed to have more outflow than vehicles in the queue. While models for this scenario have previously been presented for open-loop outflow controls, this result ensures the existence and uniqueness of solutions for the network flow dynamics in the case Lipschitz continuous feedback controllers.

Keywords— transportation networks, queuing networks, feedback control, well-posedness, reflection principle

1 Introduction

The use of dynamical flow network models to model transportation networks has recently gained a great deal of attention, see, e.g., [7, 6]. Such models describe the dynamical flow of mass among a finite set of interconnected cells have sometimes been referred to as compartmental systems in the control literature [15, 28]. Dynamical flow networks have also been used to model and design controllers for general queuing networks [16, 21, 29], where those control solutions have later been adopted to transportation networks, for example in [27, 17].

In order to capture congestion effects in transportation, such dynamical flow network systems are typically nonlinear [4]. In particular, most of them prescribe that the outflow from a cell in the network is limited by a nonlinear function of the queue lengths. In a signalized traffic network, outflows from some lanes are limited by a traffic signal, while in a highway network different traffic flow models such as the Lighthill-Whitham-Richards

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(LWR) model [19, 25] and the Cell Transmission Model (CTM) [9, 10] suggest that the flow from one partition of the highway to another is limited by the fundamental diagram. However, in signalized traffic networks, it is not always the case that there is enough vehicles to achieve the outflow limit imposed by the traffic signal controller. In this cases, first-order ODE-based models fall short of describing the network flow dynamics while guaranteeing physically meaningful properties such as mass conservation and non-negativity of queue lengths.

In this paper, we study dynamical models for deterministic queuing networks, where the actual outflow from the links may be less than what the controller allows for, in the case when there is not enough mass present on the links. The dynamic evolution of the flow network is then described by a certain differential inclusion. Under the assumption that the outflow from the links is the maximum allowed when there is mass present on the links, we investigate existence and uniqueness of solutions to the dynamical flow network.

The dynamical queuing network model we study is a particular point-queue network. This model is sometimes referred to as vertical queues, to emphasize that a possible spatial distribution of the particles queuing up is not considered in the model. Although the model is a point-queue model, congestion effects can be incorporated since the outflow from each queue is limited by feedback. Hence, it is possible to fit cell-transmission like congestion dynamics [9, 10] into the model, where the outflow from some queues are limited by a traffic signal controller, while the outflow from other queues are limited by demand and supply functions. Finite storage can then be taken into account in the model through capacity limiting supply functions.

The main contribution of this paper consists in showing that existence and uniqueness of a solution to the considered deterministic queuing networks can be guaranteed when the controller is feedback-based and Lipschitz continuous. In fact, as clarified in the paper, standard results for the existence of solutions of differential inclusions [1] do not apply in this setting. Instead, we base our arguments on a non-trivial extension of the reflection principle for reflected Brownian motion [12].

A similar point-queue model, but where the outflow controller does not have feedback, i.e., it is an open-loop controller, has been studied for a general queuing model in [18] and specifically for traffic signal control in [22] and [14]. In [22] and [14], the existence and uniqueness of a solution to the dynamics have been shown when control action is binary and pre-determined, i.e., the traffic signal at a given time point is either green or red and does not depend on the current traffic situation. However, in many models for transportation networks, the outflow is state-dependent and in many feedback based solutions for traffic signal control, one is instead considering an averaged control signal that depends continuously on the state, such as in [11, 2, 23]. In [3] it has been shown that under certain assumptions, the averaged control signal dynamics stays close to the binary control signal dynamics. While the model in [18] allows for non-binary control actions, the outflows are still assumed to be independent of the current queue lengths.

The outline of the paper is as follows: The rest of this section is devoted to introducing some basic notation. In Section 2, the deterministic queuing network model is introduced, together with examples illustrating how existing traffic models and controllers fit into the modeling framework. In Section 3 the main result of the paper is presented, existence and uniqueness of solutions for the deterministic queuing network model, and in Section 4 we discuss how the main result relates to some feedback controllers for traffic signal control. The paper is concluded by discussing some directions to further work. In the Appendix we present the proofs of some of the more technical results.

1.1 Notation

We let $\mathbb{R}_{(+)}$ denote the (non-negative) reals. For a finite set $\mathcal{A}$, we let $\mathbb{R}^{\mathcal{A}}$ denote the set of vectors with real entries indexed by the elements of $\mathcal{A}$. For a vector $a \in \mathbb{R}^n$, we let $\mathrm{diag}(a)$ in $\mathbb{R}^{n \times n}$ be a matrix with the entries of a on the diagonal and all off-diagonal entires equal to zero. With $\mathbb{1}$ we denote a vector whose all entries equals one. Inequalities between vectors are meant to hold entry-wise, i.e., e.g., $a \leq b$ for $a, b \in \mathbb{R}^{\mathcal{A}}$ means that $a_i \leq b_i$ for every i in $\mathcal{A}$. The positive part and the negative part of a vector $x \in \mathbb{R}^{\mathcal{A}}$ are denoted by $[x]_+ = \max\{x, 0\}$ in $\mathbb{R}^{\mathcal{A}}$ and $[x]_- = \max\{-x, 0\}$ in $\mathbb{R}^{\mathcal{A}}$, respectively, where max and min are applied entry-wise. Analogously, the absolute value of a vector x in $\mathbb{R}^{\mathcal{A}}$ is the vector $|x| = [x]_+ + [x]_-$ in $\mathbb{R}^{\mathcal{A}}_+$ whose entries are equal to the absolute values of the entries of x. We let $\|\cdot\|$ be the standard 2-norm, unless otherwise specified. Finally, a directed multigraph is a 4-tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \tau)$ where $\mathcal{V}$ is a finite set of nodes, $\mathcal{E}$ is a finite set of links, and $\sigma, \tau : \mathcal{E} \to \mathcal{V}$ are the maps assigning to each link i in $\mathcal{E}$ its tail node $\sigma(i)$ and head node $\tau(i)$, respectively, such that $\sigma(i) \neq \tau(i)$ for every i in $\mathcal{E}$.

2 Dynamical Model of Traffic Network

In this section, we present the dynamical model of traffic flow in a network of vertical queues and provide examples showing how existing traffic flow models and traffic signal controllers can be cast into this framework.

We model the network topology as a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \tau)$. Every link i in $\mathcal{E}$ is to be interpreted as a cell containing a time-varying queue length $x_i = x_i(t) \geq 0$, for $t \geq 0$. The state of the system is described by the vector $x = x(t)$ in $\mathcal{X} = \mathbb{R}^{\mathcal{E}}_+$ whose entries coincide with the queue lengths or sizes in the different links and evolves in continuous time as the cells exchange flow with adjacent cells and possibly with the external environment.

In particular, let each cell i in $\mathcal{E}$ possibly receive an exogenous inflow $\lambda_i = \lambda_i(t) \geq 0$ directly from the external environment. Moreover, let $z_i = z_i(t) \geq 0$ be the total outflow from cell i directed towards immediately downstream cells and possibly to the external environment. Specifically, we shall assume that a constant fraction $R_{ij} \geq 0$ of the outflow z_i from cell i is directed towards a cell $j \neq i$ such that $\tau(i) = \sigma(j)$, while the remaining part $(1 - \sum_j R_{ij}) z_i$ leaves the network directly. Conservation of mass then gives

$$\dot{x}_i = \lambda_i + \sum_{\substack{j \in \mathcal{E}: \\ \tau(j) = \sigma(i)}} R_{ji} z_j - z_i \,, \qquad i \in \mathcal{E} \,. \tag{1}$$

In order to introduce a more compact notation, let us stack the exogenous inflows in a vector $\lambda = (\lambda_i)_{i \in \mathcal{E}}$ in $\mathbb{R}^{\mathcal{E}}_+$ and the cells' outflows in a vector $z = (z_i)_{i \in \mathcal{E}}$ in $\mathbb{R}^{\mathcal{E}}_+$. Moreover, let us introduce the routing matrix R in $\mathbb{R}^{\mathcal{E} \times \mathcal{E}}_+$ whose entries $R_{ij} \geq 0$ coincide with fraction of outflow from cell i that flows directly to cell j. Observe that the network topology constraints imply that $R_{ij} = 0$ whenever $\tau(i) \neq \sigma(j)$. Moreover, conservation of mass implies that $\sum_j R_{ij} \leq 1$ for every cell i in $\mathcal{E}$, i.e., the routing matrix R has be row sub-stochastic. Equation (1) can then be expressed more compactly as

$$\dot{x} = \lambda - (I - R^T) z \,. \tag{2}$$

In order to complete the description of the dynamical flow network system, it remains to specify how the outflow vector z depends on the state vector x. In this paper, we focus on the case where the outflow z_i from a cell i is limited by a feedback-controller $\zeta_i(x)$, so that

$$0 \le z_i(t) \le \zeta_i(x(t)) , \qquad i \in \mathcal{E} , \ t \ge 0 ,$$

and that in fact the outflow z_i from cell i coincides with $\zeta_i(x)$ whenever the queue length x_i is strictly positive, i.e.,

$$x_i(\zeta_i(x(t)) - z_i) = 0 , \qquad i \in \mathcal{E} , \ t \ge 0 .$$

With this assumption, it is clear that the outflow from one link z_i is only unspecified when there are no particles present on one link, i.e., when $x_i = 0$, while the controller still gives the link service such that $\zeta_i(x) > 0$. The rationale for not forcing $z_i = \zeta_i(x)$ also when $x_i = 0$ is that in this case, if it happens also that $\lambda_i + \sum_j R_{ji} z_j < \zeta_i(x)$, then the dynamics would force the system to violate the non-negativity constraint on the queue length, i.e.,

$$x_i(t) \ge 0 , \qquad i \in \mathcal{E} , \ t \ge 0 .$$

Observe that the three constraints above may be rewritten more compactly as

$$x \ge 0 , \tag{3}$$

$$0 \le z \le \zeta(x) , \tag{4}$$

$$x^T(\zeta(x) - z) = 0 . \tag{5}$$

where $\zeta : \mathcal{X} \to \mathbb{R}_+^{\mathcal{E}}$. Although the controller is set, the dynamics above is a differential inclusion, since the outflow z_i is not uniquely specificied when for some for some cell i in $\mathcal{E}$, the queue length $x_i = 0$ but $\zeta_i(x) > 0$. As we illustrate through the examples below, situations like this can occur when a traffic signal controller serves more than one queue simultaneously when activating a phase. We shall refer to the system of differential inclusions (2)–(5) as a (feedback-controlled, deterministic) queuing network.

Throughout the paper, we assume that the routing matrix R is out-connected, meaning that for every link i in $\mathcal{E}$ there exists some link j in $\mathcal{E}$ and an integer $l \ge 0$ such that $\sum_{k \in \mathcal{E}} R_{jk} < 1$ and $(R^l)_{ij} > 0$. Under such assumption, our main result presented as Theorem 1 in Section 3 guarantees that, whenever the feedback controller $\zeta(x)$ is a Lipschitz-continuous function of the state, the queuing network (2)–(5) admits a unique solution for every initial state $x(0)$ in $\mathcal{X}$.

We conclude this section by discussing some examples in order to better motivate the considered dynamical network flow system and illustrate the usefulness of our result.

Example 1 *To illustrate how our model can be used together with a feedback based traffic signal controller, consider a small queuing network consisting of two controlled nodes as depicted in Figure 2. Assume that nodes v_1 and v_2 are equipped with two service phases, such that either the east-west or north-south going links can receive service simultaneously.*

For example, if the service allocation is computed by the Generalized Proportional Allocation (GPA) controller proposed in [23, 24], the outflow from each link will be limited by

$$\zeta_1(x) = \zeta_8(x) = \frac{x_1 + x_8}{x_1 + x_3 + x_5 + x_8 + \kappa_1} ,$$

$$\zeta_3(x) = \zeta_5(x) = \frac{x_3 + x_5}{x_1 + x_3 + x_5 + x_8 + \kappa_1} ,$$

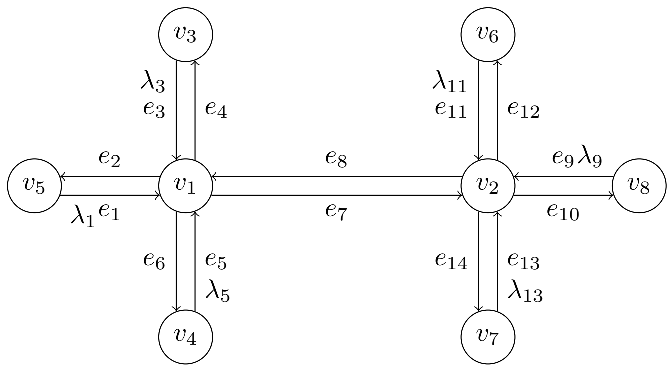

Figure 1: The network in Example 1. The network consists of two signalized junctions.

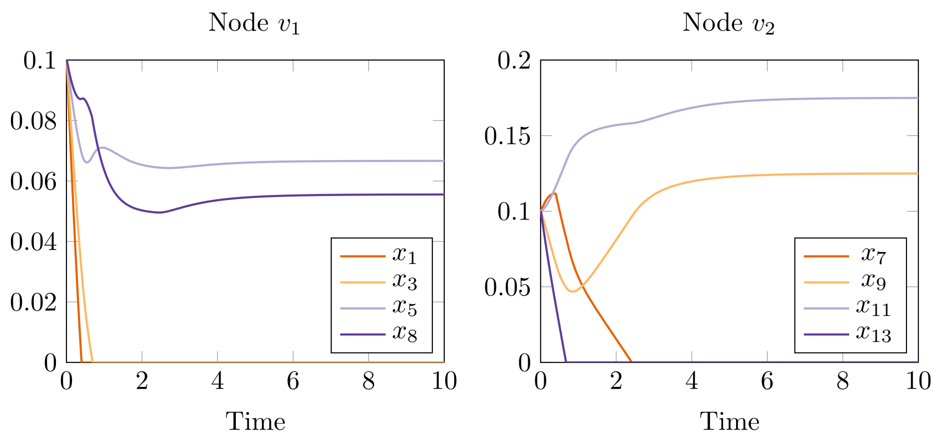

Figure 2: Time evolution of the queue lengths x_i in Example 1.

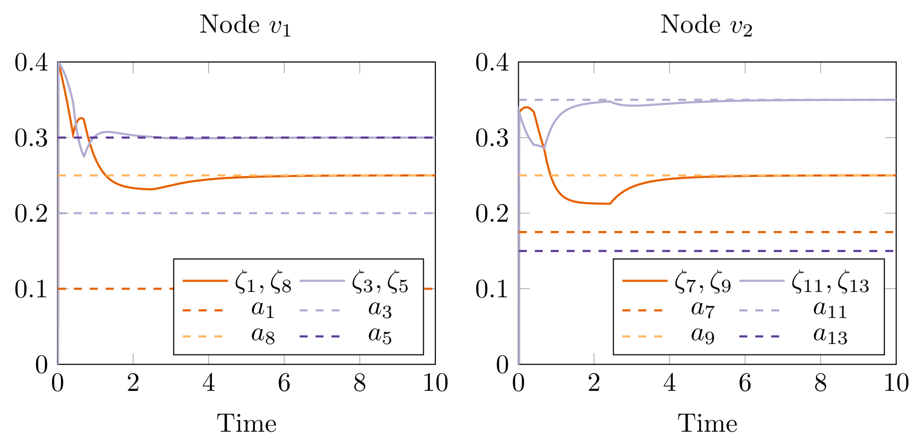

Figure 3: Time evolution of the control signals $\zeta_i(x(t))$ in Example 1. As a reference, also the equilibrium arrival rates $a_i = ((I - R^T)^{-1}\lambda)_i$ are plotted for each link i in $\mathcal{E}$.

$$\zeta_7(x) = \zeta_9(x) = \frac{x_7 + x_9}{x_7 + x_9 + x_{11} + x_{13} + \kappa_2}\,,$$

$$\zeta_{11}(x) = \zeta_{13}(x) = \frac{x_{11} + x_{13}}{x_7 + x_9 + x_{11} + x_{13} + \kappa_2}\,,$$

where $\kappa_1, \kappa_2 > 0$ are constants. The GPA-controller splits the service between the different phases in proportion to sum of the queue lengths in each each phase. The constants κ_1 and κ_2 are introduced to capture the fact that a fraction of the service cycle cannot be utilized, as there is some overhead time between the activation of subsequent phases.

For the links heading towards the boundary of the network, i.e., the links in the set $\mathcal{B} = \{e_2, e_4, e_6, e_{10}, e_{12}, e_{14}\}$, we assume that particles are allowed to flow out from the network with unit rate at all times, i.e., $\zeta_i(x) = 1$ for all i in $\mathcal{B}$. Moreover, outflow from the boundary links will leave the network, so $R_{ij} = 0$ for all i in $\mathcal{B}$ and all j in $\mathcal{E}$.

In this example, it is possible that control action is larger than the actual outflow. It can for instance happen when $x_1 = 0$, but

$$\zeta_1(x) = \frac{x_8}{x_3 + x_5 + x_8 + \kappa_1} > \lambda_1\,.$$

A numerical simulation of the queuing network is shown in Figure 2. In the simulation, we assume that $1/4$ of the inflow from each link to the nodes v_1 and v_2 is going left, $1/4$ going right, and the remaining half of the flow proceeds straight. Moreover, we let $\lambda_1 = 0.10$, $\lambda_3 = 0.20$, $\lambda_5 = 0.30$, $\lambda_9 = 0.25$, $\lambda_{11} = 0.35$, and $\lambda_{13} = 0.15$. The constants in the controllers are chosen to be $\kappa_1 = 0.1$ and $\kappa_2 = 0.2$, and all the queue lengths are initiated at 0.1, i.e., $x_i(0) = 0.1$ for every link i in $\mathcal{E}$.

In Figure 3, the control actions $\zeta_i(x)$ are plotted, together with the outflows at the equilibrium flow $z = a$. The latter can be computed as $a = (I - R^T)^{-1}\lambda$. From Figure 2 and Figure 3, we can see that controller will be equal to the equilibrium flows for all links where $x_i > 0$, while for the links where $x_i = 0$, the controller will allow for more outflow than what is physically possible, and hence $z_i < \zeta_i(x_i)$ for those links. Moreover, the control action converges to the outflows for the links with $x_i > 0$. This observation, is a consequence of the fact proven in [23] that the GPA controller is stabilizing, i.e., as long as the demands are feasible, the controller will ensure that the queue lengths stay bounded by giving each queue enough of service.

While the example above assume no propagation delay between the nodes, dynamical propagation delay can be introduced into the model by adding intermediate nodes, as the following example shows. This property makes the proposed model more adaptable to certain applications, compared to the open loop model presented in [22, 14], where the delay is assumed to be independent of the state.

Example 2 *Starting from Example 1, we introduce two intermediate nodes between the nodes v_1, and v_2, as shown in Figure 4. Moreover, we let the outflow from the added intermediate links adhere the continuous version of the Cell Transmission Model (CTM) [9, 10], used to model traffic flow in e.g. [20, 8]. To each of the links e_{15}, e_{16}, e_{17} and e_{18} we assign a demand function $d_i(x_i) \geq 0$ that is strictly increasing and Lipschitz continuous in the queue length. To each of links e_7, e_8, e_{16} and e_{18} we assign a supply function $s_i(x_i) \geq 0$ that is non-increasing and Lipschitz continuous in the queue length. The outflows from the*

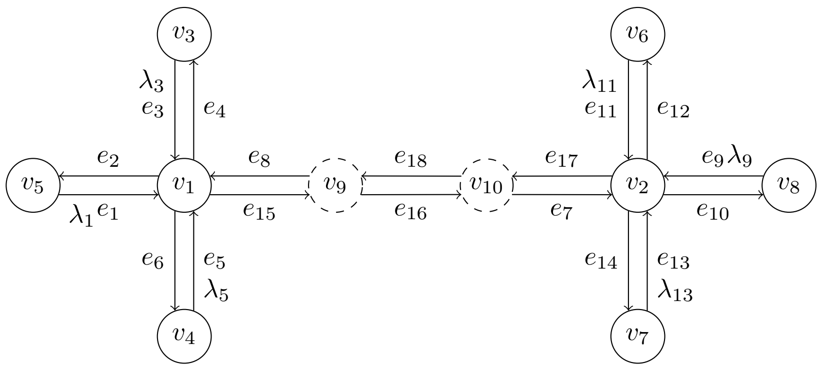

Figure 4: The network in Example 2. By introducing intermediate nodes between the junction, the flow dynamics can be discretized and a dynamic propagation delay can be modeled.

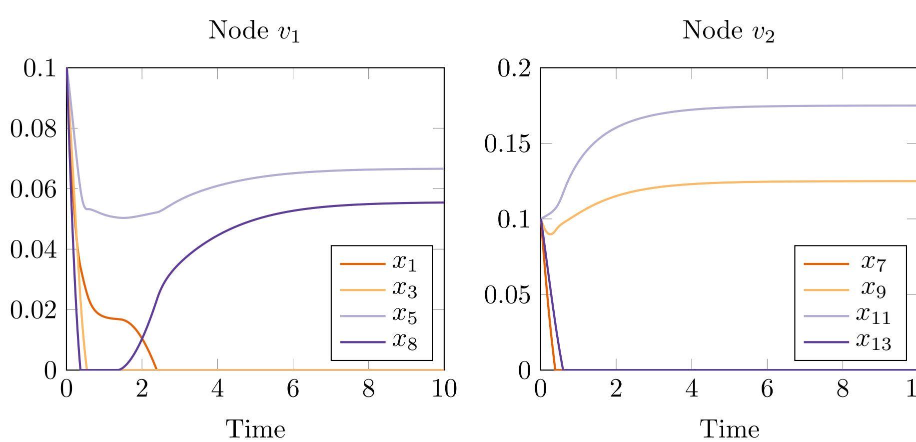

Figure 5: How the queue lengths x evolve with time in Example 2. Compared to Example 1, the trajectories becomes different, due to the flow propagation dynamics on the intermediate links.

the intermediate links are then given by

$$\zeta_{15}(x) = \min(d_{15}(x_{15}), s_{16}(x_{16}))\,, \qquad \zeta_{16}(x) = \min(d_{16}(x_{16}), s_7(x_7))\,,$$
$$\zeta_{17}(x) = \min(d_{17}(x_{17}), s_{18}(x_{18}))\,, \qquad \zeta_{18}(x) = \min(d_{18}(x_{18}), s_7(x_8))\,.$$

In Figure 5 we show the trajectories for the queue lengths on the incoming links to node v_1 and v_2. The simulation parameters are the same as in Example 1, and for all the intermediate links we let $d_i(x_i) = x_i$ and

$$s_i(x_i) = \begin{cases} 1 - x_i & \text{if } x_i \leq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we let $x_{15}(0) = x_{16}(0) = x_{17}(0) = x_{18}(0) = 0$. That the intermediate nodes introduces a propagation delay, can be seen though that it takes a longer time for the queue length on link e_8 to converge. The reason for that the same delay can not be observed on link e_7, is that the controller is already allowing for more outflow than needed, due to a long queue on link e_9.

As a remark, in the case when the demand functions are on the form

$$d_i(x_i) = C_i \frac{x_i}{x_i + \kappa_i}\,,$$

where $C_i > 0$ and $\kappa_i > 0$ are constants. If $s_j(x_j) \geq C_i$ for all x_j, then the stability analysis for the General Proportional Allocation controller, done in [23], can be applied to ensure stability of the deterministic queuing network. This since the demand function is in fact a GPA controller with just one incoming link.

3 Existence and Uniqueness of Solutions

In this section, we present a proof of existence and uniqueness of a solution to the dynamical system (2)–(5). We start off this section with an example showing why classical results about existence of solutions to differential inclusions [1] are not applicable to the analysis of the considered queuing network model (2)–(5).

Example 3 *For the purpose of illustrating why standard results for differential inclusion are not directly applicable, consider a small queuing network consisting of two parallel links between two nodes, as depicted in Figure 3. We assume that both the links belongs to the same service phase, i.e., when one link gets served, the other link gets served as well. Let the outflow controller be*

$$\zeta_1(x) = \zeta_2(x) = \frac{x_1 + x_2}{x_1 + x_2 + 1}\,. \tag{6}$$

For exogenous inflows (λ_1, λ_2), the queuing network (2)–(5) reduces to

$$\dot{x}_1 = \lambda_1 - z_1\,, \qquad \dot{x}_2 = \lambda_2 - z_2\,,$$

$$x_1 \geq 0\,, \qquad x_2 \geq 0\,,$$

$$0 \leq z_1 \leq \frac{x_1 + x_2}{x_1 + x_2 + 1}\,, \qquad 0 \leq z_2 \leq \frac{x_1 + x_2}{x_1 + x_2 + 1}\,,$$

$$x_1\left(\frac{x_1+x_2}{x_1+x_2+1}-z_1\right)=0\,,\qquad x_2\left(\frac{x_1+x_2}{x_1+x_2+1}-z_2\right)=0\,.$$

This can be rewritten more compactly as a differential inclusion

$$\dot{x}\in F(x)\,,$$

where $F(x)$ is the set-valued map defined by

$$F(x)=\begin{cases}\left\{\left(\lambda_1-\frac{x_1+x_2}{x_1+x_2+1},\lambda_2-\frac{x_1+x_2}{x_1+x_2+1}\right)\right\} & \text{if } x_1>0,\ x_2>0\,,\\ \left[\left[\lambda_1-\frac{x_2}{x_2+1}\right]_+,\lambda_1\right]\times\left\{\lambda_2-\frac{x_2}{x_2+1}\right\} & \text{if } x_1=0,\ x_2>0\,,\\ \left\{\lambda_1-\frac{x_1}{x_1+1}\right\}\times\left[\left[\lambda_2-\frac{x_1}{x_1+1}\right]_+,\lambda_2\right] & \text{if } x_1>0,\ x_2=0\,,\\ \{(\lambda_1,\lambda_2)\} & \text{if } x_1=0,\ x_2=0\,.\end{cases}$$

Now, consider the sequence of states $x^{(n)}=(1,1/n)$ in $\mathcal{X}$. Then,

$$F(x^{(n)})=\left\{y^{(n)}\right\}\,,\qquad y^{(n)}=\left(\lambda_1-\frac{1+1/n}{2+1/n},\lambda_2-\frac{1+1/n}{2+1/n}\right)\,,\qquad n=1,2\ldots\,,$$

while

$$\lim_{n\to+\infty}x^{(n)}=x^*\,,\qquad \lim_{n\to+\infty}y^{(n)}=y^*\,,$$

where

$$x^*=(1,0)\,,\qquad y^*=(\lambda_1-1/2,\lambda_2-1/2)\,,$$

and

$$F(x^*)=\{\lambda_1-1/2\}\times[[\lambda_2-1/2]_+,\lambda_2]\,.$$

Now, assume that $0<\lambda_2<1/2$, so that $[\lambda_2-1/2]_+=0$ and

$$F(x^*)=\{\lambda_1-1/2\}\times[0,\lambda_2]\,.$$

Then, clearly $y^*\notin F(x^*)$, thus implying that the set-valued map $F(x)$ is not lower semi-countinous [1] *at x^*. Moreover, chose some $\varepsilon>0$ such that $\varepsilon<1/2-\lambda_2$, and consider the neighborhood*

$$\mathcal{N}=(\lambda_1-1/2-\varepsilon,\lambda_1-1/2+\varepsilon)\times(-\varepsilon,\lambda_2+\varepsilon)$$

of $F(x^*)$. Then, since $y^{(n)}\overset{n\to+\infty}{\longrightarrow}y^*\notin\mathcal{N}$, it is immediate to verify that $y^{(n)}\notin\mathcal{N}$ for sufficiently large n, thus implying that the set-valued map $F(x)$ is not upper semi-countinous[2] *at x^*. Hence, the set-valued map $F(x)$ is neither upper nor lower semicountinous and hence the classical existence results in [1, Chapter 2] cannot be applied to the consider model of queuing network* (2)–(5).

Despite the fact that classical existence results for solutions of differential inclusions cannot be applied to the considered model of feedback controlled queuing network (2)–(5),

[1] C.f. [1, Chapter 1.1, Definition 2] and subsequent sequantial characterization of lower semi-continuity of set-valued maps.

[2] C.f. [1, Chapter 1.1, Definition 1] for the notion of upper semi-continuity of set-valued maps.

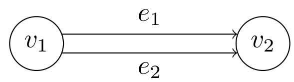

Figure 6: The small network used in Example 3.

both existence and uniqueness of solutions can proven, as formalized in the following result, which is the main contribution of this paper. Before stating it, let us define a solution of the queuing network (2)–(5) as the pair $(x(t), z(t))$ of an absolutely continuous function $x : [0, +\infty) \to \mathbb{R}_+^{\mathcal{E}}$ and a bounded measurable function $z : [0, +\infty) \to \mathbb{R}_+^{\mathcal{E}}$ that jointly satisfy (2)–(5).

Theorem 1 *Let R be an out-connected routing matrix, $\lambda : [0, +\infty) \to \mathbb{R}_+^{\mathcal{E}}$ a bounded measurable exogenous inflow vector, and $\zeta : \mathcal{X} \to \mathcal{Z}$ a Lipschitz continuous feedback controller. Then for every initial condition $x(0)$ in $\mathcal{X}$, the queuing network (2)–(5) admits a unique solution $(x(t), z(t))$.*

The proof of Theorem 1 is inspired by the reflection principle for Brownian motion, as previously presented in [12]. The main technical challenge to be faced consists in extending the analysis of [12] to the case when the outflow is determined by a Lipschitz continuous feedback controller.

Throughout the proof the of Theorem 1, we will make use of the fact that since the routing matrix is out-connected, it has a spectral radius strictly smaller than 1, see e.g. [5, Proof of Theorem 2]. Then, from [13, Lemma 5.6.10] it follows that there exists a vector norm $\|\cdot\|_\dagger$ on $\mathbb{R}^{\mathcal{E}}$ such that the induced matrix norm of R satisfies $\|R\|_\dagger < 1$. For $T > 0$, we shall consider the space $\mathcal{C}_T$ of continuous vector-valued functions $f : [0, T] \to \mathbb{R}^{\mathcal{E}}$ equipped with the norm

$$\|f\| = \left\| \sup_{0 \le t \le T} |f(t)| \right\|_\dagger .$$

Now, to a given continuous vector-valued function γ in $\mathcal{C}_T$, we associate the operator

$$\Pi_\gamma : \mathcal{C}_T \to \mathcal{C}_T$$

defined by

$$[\Pi_\gamma(v)]\,(t) = \sup_{0 \le s \le t} \left[R^T v(s) - \gamma(s) \right]_+ , \qquad 0 \le t \le T . \tag{7}$$

The following result shows that for every continuous vector-valued function γ in $\mathcal{C}_T$, the operator Π_γ admits a unique fixed point $\Psi(\gamma)$ that is a Lipschitz-continuous function of γ.

Proposition 1 *For every $T > 0$ and γ in $\mathcal{C}_T$, the operator Π_γ admits a unique fixed point*

$$\Psi(\gamma) = \Pi_\gamma(\Psi(\gamma)) \in \mathcal{C}_T . \tag{8}$$

Moreover, the operator $\Psi : \mathcal{C}_T \to \mathcal{C}_T$ that maps a continuous vector-valued function γ into the unique fixed point of the associated operator Π_γ is Lipschitz continuous.

Proposition 1 is proven in Appendix A.

Our next step towards proving Theorem 1 consists in finding an equivalent formulation of the controlled traffic network dynamics (2)–(5). Towards this goal, we introduce two operators

$$\Phi : \mathcal{C}_T \to \mathcal{C}_T \,, \qquad \Gamma : \mathcal{C}_T \to \mathcal{C}_T \,,$$

defined by

$$[\Phi(y)](t) = y(t) + (I - R^T)[\Psi(y)](t) \,, \qquad 0 \leq t \leq T \,, \tag{9}$$

and, respectively,

$$\Gamma(x)(t) = x(0) + \int_0^t \big(\lambda(s) - (I - R^T)\zeta(x(s))\big)\, \mathrm{d}s \,, \qquad : \mathcal{C}_T \to \mathcal{C}_T \,. \tag{10}$$

We will now use the operators Ψ, Φ, and Γ in (8), (9), and (10) to state another dynamical system whose solution is related to the controlled traffic network dynamics in (2)–(5).

Proposition 2 *Let R be an out-connected routing matrix, $\lambda : [0, T] \to \mathbb{R}_+^{\mathcal{E}}$ a bounded measurable exogenous inflow vector, $\zeta : \mathcal{X} \to \mathcal{Z}$ a Lipschitz continuous feedback controller, and $x(0)$ in $\mathcal{X}$. Then, $(x(t), z(t))$ is a solution of the queuing network (2)–(5) in a time interval $[0, T]$ with initial condition $x(0)$ if and only if there exist y, w in $\mathcal{C}_T$ that are absolutely continuous and such that*

$$x = \Phi(y) \,, \tag{11}$$

$$y = \Gamma(x) \,, \tag{12}$$

$$w = \Psi(y) \,, \tag{13}$$

and

$$z(t) = \zeta(x(t)) - \dot{w}(t) \,, \tag{14}$$

for almost all $0 \leq t \leq T$.

Proposition 2 is proven in Appendix B. We propose here a physical interpretation of the terms appearing in its statement. The entries of the vector $y(t)$ can be interpreted as the "signed queue lengths" on the links *if* their size were allowed to go negative, i.e, if the constraint $x \geq 0$ were removed. On the other hand, the entries of the vector $w(t)$ measure how much one must add to $y(t)$ in order to make sure that the queue lengths $x(t)$ remain non-negative for every $t \geq 0$. In Figure 7 those trajectories are illustrated for a single cell, i.e., $R = 0$. Observe that $w(t)$ is non-decreasing and only increases when $x = 0$.

We are now in a position to show how Theorem 1 follows from Proposition 1 and Proposition 2.

Proof of Theorem 1: It follows from Proposition 1 that Ψ is a Lipschitz continuous operator on $\mathcal{C}_T$. Hence, the operator Φ is Lipschitz-continuous as well and we shall denote by $\phi > 0$ its Lipschitz constant. Since $\zeta : \mathcal{X} \to \mathbb{R}^{\mathcal{E}}$ is a Lipschitz continuous function, the operator Γ is Lipschitz-continuous on $\mathcal{C}_T$ for all $T > 0$, with Lipschitz constant equal to ϖT for some constant $\varpi > 0$ that is independent from T. It then follows that, for $0 < T < (\varpi\phi)^{-1}$, the composition operator $\Phi \circ \Gamma : \mathcal{C}_T \to \mathcal{C}_T$ is Lipschitz continuous with Lipchitz constant

$$L = \varpi\phi T < 1 \,.$$

Therefore, $\Phi \circ \Gamma$ is a contraction on $\mathcal{C}_T$, hence it has a unique fixed point $x = \Phi(\Gamma(x))$. Let

$$y = \Gamma(x) \,, \qquad w = \Psi(y) \,, \qquad z = \zeta(x) - \dot{w} \,.$$

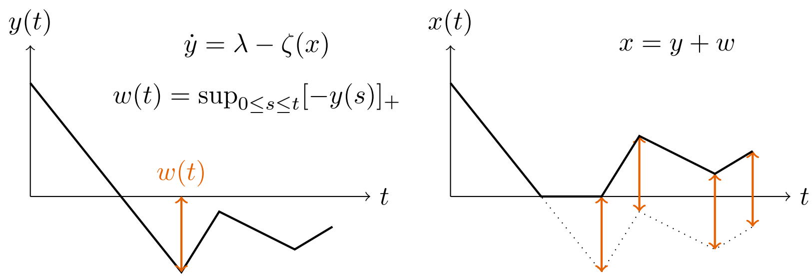

Figure 7: The connection between the quantities x, y and w in Proposition 2 for the case when the network consists of a single cell

Table 1: Comparison of different traffic signal controllers

Controller	**Feedback based**	**Lipschitz continuous wrt to queue lengths**	**Green light to empty lanes**
Fixed-Time [22]	No	No	Yes
GPA [24]	Yes	Yes	Yes
MaxPressure [27]	Yes	No	Yes, but discrete time dynamics
Cyclic BackPressure [17]	Yes	Yes (for fluid approx.)	Yes (for fluid approx.)
Gramian Based [2]	Yes	Unknown	Yes, but proportional outflow
Averaged Model [11]	Yes	Unknown	Yes, but mapped through a demand function

By Proposition 2 we get that such (x, z) is the unique solution to the queuing network (2)–(5) on $[0, T]$ with initial condition $x(0)$. Existence and uniqueness of the solution $(x(t), z(t))$ of the queuing network (2)–(5) can then be extended to every $t \geq 0$ by virtue of standard continuation arguments. ∎

4 Discussion

The result presented in Theorem 1 is useful when the transportation network is modeled through continuous-time dynamics, and the outflows from the queues are controlled with a Lipschitz continuous feedback controller. Moreover, for the dynamical system (2)–(5) to be effectively differential inclusion —rather than a differential equation— there must be a possibility for the controller to give service to empty queues. This situation can, for instance, occur when more than one queue belongs to a service phase, but also if one queue receives service longer than needed to empty the queue.

In Table 1 we summarize how the presented result relates to some previously proposed models and policies for traffic signal control. One common way to address the modeling issue when empty queues receive green light, is to introduce a demand function that models the outflow from the queue. The outflow from one queue i in $\mathcal{E}$ is then $z_i = d_i(x_i)\zeta_i(x)$

where $d_i(x_i) \geq 0$ is the demand function. The demand function has then to be chosen such that $d_i(0) = 0$. In [2] the demand function is assumed to be linear, while [11] allows for more general demand functions. While this modeling approach ensures that the queue lengths can not go below zero even when the traffic signal controller $\zeta_i(x)$ is strictly positive, the modeling approach also deviates from the point-queue dynamics.

In point-queue models, the traffic will flow out with maximum capacity when it is allowed to do so. In contrast, by introducing a demand function, the outflow will increase with the queue lengths. Point-queue models have previously also been used when developing control strategies for traffic signal control, such as the MaxPressure [27] and a point-queue model is also used for the analysis of fixed-time controllers in [22]. The assumption that the outflow will be at maximum capacity, but where this capacity may depend on which phase that is activated when vehicles are queuing up, is also in line with the classical approaches to model a signalized junctions [26].

On the other hand, models where the traffic flow depends on the number of vehicles, are common to model traffic propagation between junctions. For example, in [11] a continuous version of the Cell Transmission Model [9, 10] is used. As illustrated in Example 2, the model analyzed in this paper allows for incorporating such dynamics between the junctions. Hence, the presented model makes it possible to combine models for traffic flow propagation between the junctions with point-queue dynamics at the junctions.

5 Conclusion

In this paper, we have presented a dynamical model for transportation networks consisting in a deterministic queuing network, where a feedback-controller limits the outflow from each link. The feedback-controller may allow for more outflow that is physically possible to flow out. Due to this property, the queuing network is described as a differential inclusion. We show that such differential inclusion admits a unique solution for every initial state. In the future, we plan to extend the well-posedness results, at least for the existence part, to non-Lipschitz and possibly discontinuous feedback controls as those mentioned in [24]. It would also be of great interest to study the case of time-varying routing matrix R and/or to analyze how robust the model is to the choice of such matrix.

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A Proof of Proposition 1

As a first step in proving Proposition 1, we introduce the following the following lemma.

Lemma 1 *Let R be an out-connected routing matrix, $\lambda : [0,T] \to \mathbb{R}_+^{\mathcal{E}}$ a bounded measurable exogenous inflow vector, and $\zeta : \mathcal{X} \to \mathcal{Z}$ a Lipschitz continuous feedback controller. For every $T > 0$ and every continuous vector-valued function γ in $\mathcal{C}_T$, the operator Π_γ is a contraction on $\mathcal{C}_T$.*

Proof We will first show that Π_γ is a contraction on $\mathcal{C}_T$. For any v, w in $\mathcal{C}_T$, $0 \le s \le t \le T$, and i in $\mathcal{E}$, put

$$f(s) = [R^T v(s) - \gamma(s)]_i \,, \quad g(s) = [R^T w(s) - \gamma(s)]_i \,, \tag{15}$$

$$h(s) = f(s) - g(s)\,.$$

Choose some

$$s^* \in \arg\max_{0\le s\le t}[f(s)]_+\,, \qquad q^* \in \arg\max_{0\le s\le t}[g(s)]_+\,,$$

and observe that

$$[f(s^*)]_+ = [g(s^*) + h(s^*))]_+ \le [g(s^*)]_+ + [h(s^*))]_+\,, \tag{16}$$

$$[g(q^*)]_+ = [f(q^*) - h(q^*))]_+ \le [f(q^*)]_+ + [-h(q^*))]_+ = [f(q^*)]_+ + [h(q^*))]_-\,. \tag{17}$$

Using (16) and the fact that $[f(s^*)]_+ = \sup_{0\le s\le t}[f(s)]_+$, we get

$$\begin{aligned}\sup_{0\le s\le t}[h(s)]_+ &\ge [h(s^*)]_+\\ &\ge [f(s^*)]_+ - [g(s^*)]_+\\ &\ge \sup_{0\le s\le t}[f(s)]_+ - \sup_{0\le s\le t}[g(s)]_+\,.\end{aligned}$$

Analogously, (17) and the fact that $[g(q^*)]_+ = \sup_{0\le s\le t}[g(s)]_+$ give

$$\begin{aligned}\sup_{0\le s\le t}[h(s)]_- &= \sup_{0\le s\le t}[-h(s)]_+\\ &\ge [g(q^*)]_+ - [f(q^*)]_+\\ &\ge \sup_{0\le s\le t}[g(s)]_+ - \sup_{0\le s\le t}[f(s)]_+\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{0\le s\le t}|h(s)| &= \max\left\{\sup_{0\le s\le t}[h(s)]_+, \sup_{0\le s\le t}[h(s)]_-\right\}\\ &\ge \left|\sup_{0\le s\le t}[f(s)]_+ - \sup_{0\le s\le t}[g(s)]_+\right|\,.\end{aligned}$$

Now, let us define the vector α in $\mathbb{R}^{\mathcal{E}}$ with entries

$$\alpha_i = \sup_{0\le t\le T}\left|\left[\Pi_\gamma(v)\right]_i(t) - \left[\Pi_\gamma(w)\right]_i(t)\right|\,,$$

for all i in $\mathcal{E}$. Using (7) and (15), we get

$$\begin{aligned}\alpha_i &= \sup_{0\le t\le T}\left|\sup_{0\le s\le t}[f(s)]_+ - \sup_{0\le s\le t}[g(s)]_+\right|\\ &\le \sup_{0\le t\le T}\sup_{0\le s\le t}|h(s)|\\ &= \sup_{0\le t\le T}\left|[R^T(v(t)-w(t))]_i\right|\\ &\le \sum_j R_{ji}\sup_{0\le t\le T}|v_j(t)-w_j(t)|\,.\end{aligned}$$

Hence,

$$\|\Pi_\gamma v - \Pi_\gamma w\| = \|\alpha\|_\dagger \le \left\|R^T\right\|_\dagger \|v-w\|\,.$$

Since $\left\|R^T\right\|_\dagger < 1$, the above proves that Π_γ is a contraction on $\mathcal{C}_T$. ■

We are now ready to prove Proposition 1.

Proof of Proposition 1 It follows from Lemma 1 and the Banach fixed point theorem that for every continuous vector-valued function γ in $\mathcal{C}_T$, the operator Π_γ admits a unique fixed point $\Psi(\gamma) = \Pi_\gamma(\Psi(\gamma))$ in $\mathcal{C}_T$.

To prove the second part of the proposition, for $k \geq 0$, let Π_γ^k be the composition of Π_γ with itself k times. Fix three functions v, γ, η in $\mathcal{C}_T$ and for $0 \leq t \leq T$, define

$$\delta^k(t) = \left[\Pi_\gamma^k(v)\right](t) - \left[\Pi_\eta^k(v)\right](t)\,.$$

Then, we have that

$$\begin{aligned}
\left|\delta^{k+1}(t)\right| &= \left|\left[\Pi_\gamma^{k+1}(v)\right](t) - \left[\Pi_\eta^{k+1}(v)\right](t)\right| \\
&= \left|\sup_{0\leq s\leq t}\left[R^T[\Pi_\gamma^k v](s) - \gamma(s)\right]_+ - \sup_{0\leq s\leq t}\left[R^T[\Pi_\eta^k v](s) - \eta(s)\right]_+\right| \\
&\leq \left|\sup_{0\leq s\leq t}\left[R^T\left([\Pi_\gamma^k v](s) - [\Pi_\eta^k v](s)\right) - (\gamma(s) - \eta(s))\right]_+\right| \\
&\leq \sup_{0\leq s\leq t}\left|R^T\delta^k(s)\right| + \sup_{0\leq s\leq t}|\gamma(s) - \eta(s)|
\end{aligned}$$

so that

$$||\delta^{k+1}|| \leq \left\|R^T\right\|_\dagger \left\|\delta^k\right\| + \|\gamma - \eta\|\,.$$

It follows that, for all v in $\mathcal{C}_T$ and $k \geq 0$,

$$\left\|\Pi_\gamma^k(v) - \Pi_\eta^k(v)\right\| \leq \sum_{l=0}^{k}\left\|R^T\right\|_\dagger^l \|\gamma - \eta\|\,.$$

Since $||R^T||_\dagger < 1$ and Π_γ and Π_η are both contractions with fixed points $\Psi(\gamma)$ and $\Psi(\eta)$, respectively, taking the limit as k grows large in the above gives

$$\begin{aligned}
\|\Psi(\gamma) - \Psi(\eta)\| &= \lim_{k\to\infty}\left\|\Pi_\gamma^k(v) - \Pi_\eta^k(v)\right\| \\
&\leq \sum_{l=0}^{+\infty}\left\|R^T\right\|_\dagger^l \|\gamma - \eta\| \\
&= \frac{\|\gamma - \eta\|}{1 - \|R^T\|_\dagger}\,.
\end{aligned}$$

which concludes the proof of Proposition 1. ■

B Proof of Proposition 2

(i) Let $(x(t), z(t))$ be a solution of the controlled traffic network dynamics (2)–(5) on $[0, T]$ with initial condition $x(0)$. For $0 \leq t \leq T$, let

$$w(t) = \int_0^t (\zeta(x(s)) - z(s))\mathrm{d}s\,, \tag{18}$$

$$y(t) = x(t) - (I - R^T)w(t)\,. \tag{19}$$

We will show that (11)–(14) are satisfied. Indeed, taking the time derivative of both sides of (18) gives (14). On the other hand, (19), (2), and (18) yield

$$\begin{aligned} y(t) &= x(t) - (I - R^T)w(t) \\ &= x(0) + \int_0^t (\lambda(s) - (I - R^T)z(s))\mathrm{d}s - (I - R^T)w(t) \\ &= x(0) + \int_0^t (\lambda(s) - (I - R^T)\zeta(x(s)))\mathrm{d}s \\ &= \Gamma(x)(t)\,, \end{aligned}$$

so that (12) is satisfied as well. Moreover, (19) and (13) clearly imply (11). Hence, it remains to prove (13). For that, first observe that (18) and (5) imply that

$$x \geq 0\,, \qquad x^T\dot{w} = 0\,, \qquad 0 \leq \dot{w} \leq \zeta(x)\,. \tag{20}$$

In turn, the above and (19) imply that, for $0 \leq s \leq t$,

$$w(t) \geq w(s) = R^T w(s) + x(s) - y(s) \geq R^T w(s) - y(s)\,,$$

so that

$$w(t) \geq \sup_{0 \leq s \leq t} \left\{ R^T w(s) - y(s) \right\}\,.$$

Since $w(0)$ is non-increasing and $w(0) = 0$, we have $w(t) \geq 0$, which together with the above gives

$$w(t) \geq \sup_{0 \leq s \leq t} \left[R^T w(s) - y(s) \right]_+ = \Pi_y(w)(t)\,.$$

In fact, if the above were not an identity for some $0 \leq t \leq T$, there would exist some $0 \leq t^* \leq T$ and i in $\mathcal{E}$ such that

$$w_i(t^*) > \sup_{0 \leq s \leq t^*} \left\{ \sum\nolimits_j R_{ji}w_j(s) - y_i(s) \right\}\,, \qquad \dot{w}_i(t^*) > 0\,. \tag{21}$$

But the second inequality above and (20) imply that $x_i(t^*) = 0$ so that, by (19),

$$y_i(t^*) = \sum\nolimits_j R_{ji}w_j(t^*) - y_i(t^*)\,,$$

which contradicts (21). Hence, we necessarily have

$$w(t) = \Pi_y(w)(t)\,, \qquad 0 \leq t \leq T\,,$$

i.e., w is the fixed point Π_y on $\mathcal{C}_T$, so that (13) is satisfied.

(ii) Let $w, x, y, z \in \mathcal{C}_T$ be such that y and w are absolutely continuous and (11)–(14) are satisfied. Then, for $0 \le t \le T$, an application of (11), (9), (12), (13), (10), and (14) give

$$\begin{aligned} x(t) &= \Phi(y)(t) \\ &= y(t) + (I - R^T)\Psi(y)(t) \\ &= \Gamma(x)(t) + (I - R^T)w(t) \\ &= x(0) + \int_0^t \big(\lambda(s) - (I - R^T)\zeta(x(s))\big)\,\mathrm{d}s + (I - R^T)\int_0^t (\zeta(x(s)) - z(s))\,\mathrm{d}s \\ &= x(0) + \int_0^t \big(\lambda(s) - (I - R^T)z(s)\big)\,\mathrm{d}s\,, \end{aligned}$$

hence (2) is satisfied. On the other hand, (11), (9), (8), and (7) give

$$\begin{aligned} x(t) &= \Phi(y)(t) \\ &= y(t) + (I - R^T)\Psi(y)(t) \\ &= y(t) - R^T\Psi(y)(t) + \sup_{0\le s\le t}\big[R^T\Psi(y)(s) - y(s)\big]_+ \\ &\ge 0\,. \end{aligned} \tag{22}$$

Moreover, (13), (8), and (7) yield

$$w(t) = \Psi(y)(t) = \sup_{0\le s\le t}\big[R^T w(s) - y(s)\big]_+\,, \tag{23}$$

so that $w_i(t)$ is non-decreasing for all i in $\mathcal{E}$, hence $\dot{w} \ge 0$. Furthermore, let $\mathcal{I} := \{i \in \mathcal{E} : \dot{w}_i(t) > 0\}$ be the set of cells i such that $w_i(t)$ is strictly increasing at time t. It then follows from (23) that

$$w_i(t) = \sum_{j\in\mathcal{E}} R_{ji}w_j(t) - y_i(t)\,, \qquad i \in \mathcal{I}\,. \tag{24}$$

Equation (24) implies that, for $i \in \mathcal{I}$,

$$\begin{aligned} \dot{w}_i(t) &= \sum_{j\in\mathcal{E}} R_{ji}\dot{w}_j(t) - \dot{y}_i(t) \\ &= \sum_{j\in\mathcal{I}} R_{ji}\dot{w}_j(t) - \lambda_i(t) + \zeta_i(x(t)) - \sum_{j\in\mathcal{E}} R_{ji}\zeta_j(x(t)) \\ &\le \sum_{j\in\mathcal{I}} R_{ji}\dot{w}_j(t) - \lambda_i(t) + \zeta_i(x(t)) - \sum_{j\in\mathcal{I}} R_{ji}\zeta_j(x(t))\,. \end{aligned}$$

The above implies that

$$(I - R_{\mathcal{I}\mathcal{I}}^T)\dot{w}_{\mathcal{I}}(t) \le (I - R_{\mathcal{I}\mathcal{I}}^T)\zeta_{\mathcal{I}}(x(t)) - \lambda_{\mathcal{I}}(t)\,, \tag{25}$$

where $R_{\mathcal{I}\mathcal{I}}$ is the $\mathcal{I} \times \mathcal{I}$ block of R and $\dot{w}_{\mathcal{I}}(t)$, $\zeta_{\mathcal{I}}(x(t))$, and $\lambda_{\mathcal{I}}(t)$ are the $\mathcal{I}$ blocks of the corresponding vectors $\dot{w}(t)$, $\zeta_{\mathcal{I}}(x(t))$, and $\lambda_{\mathcal{I}}(t)$. Since R is out-connected, each of its diagonal blocks such as $R_{\mathcal{I}\mathcal{I}}$ has spectral radius smaller than 1. Hence $(I - R_{\mathcal{I}\mathcal{I}}^T)$ invertible with nonnegative inverse $(I - R_{\mathcal{I}\mathcal{I}}^T)^{-1}$. Hence, (25) implies that

$$\dot{w}_{\mathcal{I}}(t) \le \zeta_{\mathcal{I}}(x(t)) - (I - R_{\mathcal{I}\mathcal{I}}^T)^{-1}\lambda_{\mathcal{I}}(t) \le \zeta_{\mathcal{I}}(x(t))\,.$$

Since $\dot{w}_{\mathcal{E}\setminus\mathcal{I}}(t) = 0$ by definition and we have already noticed that $\dot{w}(t) \geq 0$, we thus have that $z = \zeta(x) - \dot{w}$ satisfies

$$0 \leq z \leq \zeta(x)\,. \tag{26}$$

Finally, using again (11), (9), (13), and (24), one gets that

$$x_i(t) = y_i(t) + w_i(t) - \sum_j R_{ji} w_j(t) = 0$$

for every $i \in \mathcal{I}$. Along with (22) and (26), this implies that

$$x^T(\zeta(x) - z) = x^T \dot{w} = 0\,. \tag{27}$$

From (22), (26), and (27) it follows that (5) is satisfied. Therefore (x, z) is a solution of (2)–(5). ■